Thin points of a class of Markov processes with jumps

Xiaochuan YANG, Université Paris-Est

joint work with S. Seuret (Créteil, France)

Probabilités de demain

IHES, Paris

17/5/2016

RDMath IdF
Domaine d'Intérêt Majeur (DIM) en Mathématiques

îledeFrance

Basic problem

- $X = \{X_t, t \in [0,1]\}$ in $\mathbb{R}^d$.
- $\mu(A) = \int_0^1 \mathbf{1}_A(X_t)dt$ for all $A \subset \mathbb{R}^d$.

Question : regularity ?

Basic problem

- $X = \{X_t, t \in [0,1]\}$ in $\mathbb{R}^d$.
- $\mu(A) = \int_0^1 \mathbf{1}_A(X_t)dt$ for all $A \subset \mathbb{R}^d$.

Question : regularity ?

- Absolute continuity (local times when X is Markovian).
- Local dimensions, i.e. for $x \in \text{supp}(\mu)$, the positive real h such that

$$\mu(B(x,r)) \sim r^h.$$

  - NOT always well defined : $\lim_{r\to 0} \frac{\ln \mu(B(x,r))}{\ln r}$ may not exist.
  - How does h depend on the value x ? described via a regularity exponent $h(x)$.

Examples

B : Brownian motion in $\mathbb{R}^d$.

- $d = 1$: local times exist [Lévy].
- $d \geq 2$: local dimension is 2 for all $x \in \mathrm{supp}(\mu)$ [Perkins-Taylor].

Examples

B : Brownian motion in $\mathbb{R}^d$.

- $d = 1$: local times exist [Lévy].
- $d \geq 2$: local dimension is 2 for all $x \in \text{supp}(\mu)$ [Perkins-Taylor].

σ : α-stable subordinator, i.e. increasing stable Lévy process in $\mathbb{R}^+$.

- Local dimension is α for μ-almost every $x \in \text{supp}(\mu)$ [Hu-Taylor].
- Exceptional points ? Yes.

Examples

B : Brownian motion in $\mathbb{R}^d$.

- $d = 1$: local times exist [Lévy].
- $d \geq 2$: local dimension is 2 for all $x \in \text{supp}(\mu)$ [Perkins-Taylor].

σ : α-stable subordinator, i.e. increasing stable Lévy process in $\mathbb{R}^+$.

- Local dimension is α for μ-almost every $x \in \text{supp}(\mu)$ [Hu-Taylor].
- Exceptional points ? Yes.

Related question :

For B in high dimension, one might wonder how the local regularity of μ fluctuates in logarithmic order, see Dembo-Peres-Rosen-Zeitouni.

Framework : multifractal analysis

Goal : distinguish different local behaviors of μ by a description of the "size" of the set of points with given regularity.

Definition

The upper local dimension of μ at x is defined by

$$\overline{h}(\mu, x) = \limsup_{r \to 0} \frac{\ln \mu(B(x,r))}{\ln r}.$$

One defines similarly the lower local dimension $\underline{h}(\mu, x)$ and local dimension $h(\mu, x)$ when the limit exists.

Definition

Define the iso-holder sets

$$\overline{E}(h) = \{x \in \operatorname{supp}(\mu) : \overline{h}(\mu, x) = h\}.$$

The upper multifractal spectrum of μ is the mapping

$$\overline{d}_\mu(\cdot) : h \mapsto \dim_{\mathcal{H}} \overline{E}(h).$$

One defines similarly $\underline{d}_\mu(\cdot)$ and $d_\mu(\cdot)$.

"Recall" : Hausdorff dimension describes the size of "small" sets in a metric space, e.g. a triadic Cantor set in $\mathbb{R}^1$.

Thin points for α-stable subordinator

Recall local dimension exists for typical points : $h(\mu, x) = \alpha$ for μ-almost every point in $\operatorname{supp} \mu$, i.e.

$$\mu(B(x,r)) \sim r^{\alpha}.$$

However, there are "many" points with smaller than normal mass, i.e.

$$\mu(B(x,r)) \sim r^{h} \text{ with } h > \alpha.$$

They are called thin points.

Theorem (Hu-Taylor)

A.s. the following holds

$$\overline{d}_{\mu}(h) = \begin{cases} \alpha(\frac{2\alpha}{h} - 1) & \text{if } h \in [\alpha, 2\alpha], \\ -\infty & \text{otherwise.} \end{cases}$$

Our process : stable-like jump diffusion

- **Goal :** describe thin points of jump diffusions (i.e. jumping SDE) by multifractal analysis.
- **Difference/Difficulty :** no more stationary increment, Markovian dynamic is space-dependent.

Definition (Bass)

The stable-like jump diffusion is a Markov processes with generator

$$\mathcal{L}f(x) = \int_0^1 f(x+u) - f(x) \frac{\beta(x)du}{u^{1+\beta(x)}}$$

where β is a Lipschitz function taking value in $[\varepsilon, 1-\varepsilon]$.

Remark : when $\beta(\cdot) = \alpha \in (0,1)$, one recovers α-stable subordinator (truncated large jumps).

The stable-like jump diffusion satisfies the jumping SDE

$$M_t = \int_0^t \int_0^1 z^{1/\beta(M_{s-})} N(ds, dz).$$

where $N(ds, dz)$ is a Poisson random measure with intensity $\pi(dz) = dz/z^2$.

Remind : dimension of the sets $\overline{E}(h) = \{x \in \text{supp}(\mu) : \overline{h}(\mu, x) = h\}$. Need a translation !

Preparation I : heuristic computation

If $x = M_t \in \overline{E}(h)$, necessarily

$$\mu(B(M_t, r_n)) \leq {r_n}^{h-\varepsilon}, \text{ for } r_n \to 0.$$

$\mu(\cdot)$ measures the time spent by M inside balls, last inequality means M can not move too slowly, precisely, infinitely often

$$|M_{t+2^{-n}} - M_t| \wedge |M_t - M_{t-2^{-n}}| \geq 2^{-n/((h-\varepsilon)\beta(M_t))}.$$

Need estimate for increments.

Preparation II : a key estimate

Proposition

For all $\delta > 1$, $m \in \mathbb{N}^*$, *with probability larger than* $1 - e^{-m}$, *for* $|t-s| \sim 2^{-m}$,

$$\left| \int_s^t \int_0^{2^{-\frac{m}{\delta}}} z^{1/\beta(M_{u-})} N(du, dz) \right| \leq \log\left(\frac{1}{|s-t|}\right)^2 |s-t|^{\frac{1}{\delta \cdot \widehat{\beta}^m_{s,t}}}$$

with $\widehat{\beta}^m_{s,t} \approx \sup_{u \in [s,t]} \beta(M_{u-})$.

Remark : uniformly, small jumps accumulation has the same effect of a single jump.

So there are two “large” jumps beside t satisfying $M_t \in \overline{E}(h)$ for infinitely many time scales.

Highlight double jumps configuration in the Poisson point process gives an upper bound for

$$\dim_{\mathcal{H}}\{t : \overline{h}(\mu, M_t) = h\}.$$

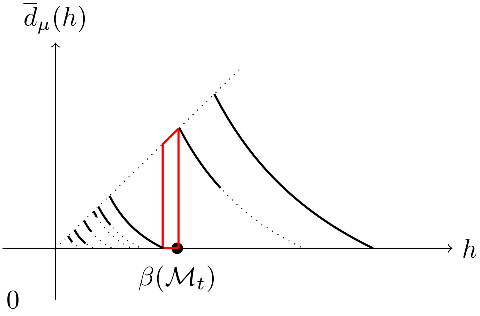
From time to space : still need an analog of the dimension doubling theorem for BM

Lower bound is more involved (construction of Cantor sets inside iso-Holder sets).

Multifractal spectrum

Theorem (16' Seuret and Y.)

A.s. the upper multifractal spectrum of μ is



Remark : superposition of random curves.

Merci de votre attention !