Unit volume Liouville measure on the sphere with (γ, γ, γ)-insertions: the link between two constructions

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Two constructions of random measures on the sphere by David°, Duplantier•, Kupiainen°, Miller•, Rhodes°, Sheffield•, Vargas°.
$\circ = [DKRV14]$: explicit formulæ for correlation functions, $n \geq 3$ insertions of arbitrary weights, suitable for compact surfaces of all genus.
$\bullet = [DMS14]$: $n \leq 2$ insertions with same weight, metric in the $\gamma = \sqrt{8/3}$ case, SLE/GFF coupling, suitable for non-compact surfaces.

Goal of $[AHS15]$: find a link between these two constructions.

Outline

Section I

Conformal embedding

Möbius transformations

as the automorphism group of the Riemann sphere

Definition

A (conformal) automorphism φ of the complex plane $\mathbb{C}$ writes

$$\varphi : z \mapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$.

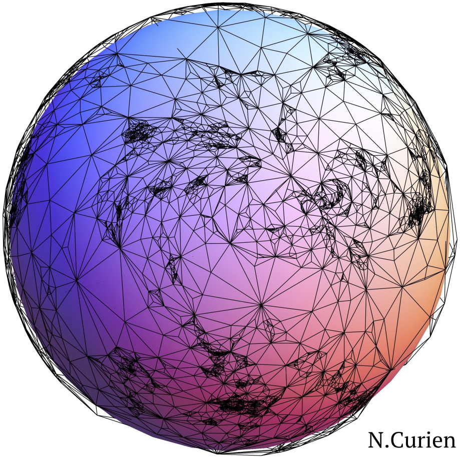
Exercice 1: give all φ such that $\varphi(0) = 0, \varphi(1) = 1, \varphi(\infty) = \infty$.
Exercice 2: give all φ such that $\varphi(0) = 0, \varphi(\infty) = \infty$.

Embedding with three marked points

as a well-defined random measure

Take a large random planar map with chosen marked points (z_1, z_2, z_3) and some conformal structure.
We "embed" this map on the sphere by sending conformally (z_1, z_2, z_3) to $(0, 1, \infty)$.
There is a unique way to do it – the limiting measure should be described by a random measure.

Conjecture: choose the three marked points uniformly among all vertices, convergence to Liouville measure with three insertions of weight γ.



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Embedding with two marked points

as an equivalence class of random measures

Imagine instead we only consider two points (z_1, z_2) and we map them to $(0, \infty)$. The mapping is ill-defined! We make use of the following equivalence class:

Definition (A quotient space Q)

Two (random) measures with marked points $(D, \mu, s_1, \ldots, s_n)$ and $(D', \nu, t_1, \ldots, t_n)$ are said equivalent if there is a (random) conformal map φ from D to D' that maps $(s_1, \ldots, s_n)$ to $(t_1, \ldots, t_n)$ and such that $\varphi_*(\mu) = \nu$; φ_* is the pushforward defined by $\varphi_*(\mu)(A) = \mu(\varphi^{-1}(A))$.

In particular, if we fix $\mathbb{C}$ with two marked points $(0, \infty)$, we get a family of (random) measures defined *modulo a dilatation*. One should describe this limit using a construction that is not sensible to the action of a certain subgroup of the Möbius group (here, the dilatations).

Section II

Two constructions

The DKRV definition

of the unit volume Liouville measure with $n \geq 3$ insertions

Definition (Unit volume Liouville measure)

Let g be a metric on the sphere. Let X_g be a whole plane GFF such that $\int_{\mathbb{R}^2} X_g(z)dg = 0$. Consider

$$X_L = X_g(z) + \sum_i \alpha_i \ln|z - z_i|$$

and let $Z_\gamma(\mathbb{R}^2) = \int_{\mathbb{R}^2} e^{\gamma X_L(z)} d\lambda_g$ the volume form associated with X_L.
The law of the unit volume Liouville measure is given by

$$\mu(A) = \int_A e^{\gamma X_U(z)} d\lambda_g$$

where $X_U = X_L - \frac{1}{\gamma} \ln Z_\gamma(\mathbb{R}^2)$ under the measure $Z_\gamma(\mathbb{R}^2)^{\frac{2Q - \sum_i \alpha_i}{\gamma}} d\mathbb{P}_{X_g}$.

The DMS equivalence class of random measures

with two γ-insertions at 0 and ∞

Definition (Bessel process encoding)

Every distribution on $\mathbb{C}$ can be decomposed into two parts:
– the radial part: average on circles $\partial B(0, r)$;
– the lateral noise part: fluctuation on each circles.

Let $\delta = 4 - 8/\gamma^2$ and ν_δ^{BES} the Bessel excursion measure of dimension δ.
We sample the radial part R in the following way:
1. Sample a Bessel excursion $\mathfrak{e}$ w.r.t. ν_δ^{BES};
2. Reparametrizing $\frac{1}{\gamma} \log \mathfrak{e}$ to have unit quadratic variation.

Add (independently) the lateral noise N part by projection.

This will give us a distribution (in fact, a Gaussian field) *defined modulo dilatation*. Take the exponential: we get the equivalence class of random measures with two γ-insertions at 0 and ∞.

Section III

Theorem and consequences

Main theorem of [AHS15]

From DMS14 to DKRV14

For better comprehension, we state the theorem in plain words.

Theorem (AHS15)

Take the sphere, or the whole plane.

1. *Consider a measure in the DMS equivalence class with two γ-insertions at 0 and ∞;*
2. *Choose a third point z w.r.t. this measure;*
3. *Use a conformal map that shifts $(0, z, \infty)$ to $(0, 1, \infty)$;*
4. *Push-forward the chosen measure in the DMS class by this conformal map;*
5. *We get DKRV measure with three γ-insertions at 0, 1 and ∞!*

Attention! It is not trivial to describe the random conformal map in step 3.

Consequence

From DKRV14 to DMS14...

Remark (Consequence)

1. *Take DKRV measure with three* γ*-insertions at* 0, 1 *and* ∞;
2. *Forget about the point* 1, *and pass to the quotient space* Q;
3. *We get the DMS equivalence class with two* γ*-insertions at* 0 *and* ∞.

Thanks!

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