Bismut-Elworthy-Li formulae for Bessel processes

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Motivation

We are interested in the regularity properties of the semigroup of solutions to some SPDEs (or SDEs) with a *singular* drift. Indeed, quite often, the semigroup might have continuity properties (e.g. strong Feller property) that can be exploited to deduce interesting information on the underlying process.

Problem: How to derive strong Feller property for the semigroup in a non-dissipative regime ?

Plan:

1. The strong Feller property
2. The Bismut-Elworthy-Li formula
3. The case of Bessel processes

The strong Feller property

Let E be a Polish space, and $(P_t)_{t\geq 0}$ a Markovian semi-group over E. Consider the spaces of functions:

$$B_b(E) := \{\varphi : E \to \mathbb{R} \text{ bounded and Borel}\},$$

$$C_b(E) := \{\varphi : E \to \mathbb{R} \text{ bounded and continuous}\}.$$

Definition

$(P_t)_{t\geq}$ satisfies the Feller property if, for all $t \geq 0$ and $\varphi \in C_b(E)$, $P_t\varphi \in C_b(E)$.
$(P_t)_{t\geq}$ satisfies the strong Feller property if, for all $t > 0$ and $\varphi \in B_b(E)$, $P_t\varphi \in C_b(E)$.

In the sequel we consider $E = \mathbb{R}$ and $(P_t)_{t\geq 0}$ on $\mathbb{R}$ given by:

$$(P_t\varphi)(x) := \mathbb{E}\left[\varphi\left(X_t(x)\right)\right] \tag{1}$$

where $(X_t(x))_{t\geq 0}$ is solution to the SDE:

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dB_t \\ X_0 &= x, \end{cases}$$

where B is a Brownian motion in $\mathbb{R}$ and $b, \sigma : \mathbb{R} \to \mathbb{R}$ are smooth.

Example (Drift)

Suppose $b = 1$ *and* $\sigma = 0$, *then* $P_t\varphi(x) = \varphi(x+t)$ *for all* $\varphi \in B_b(E)$.
$(P_t)_{t\geq 0}$ *has the Feller property, but not the strong Feller one.*

Example (Brownian motion)

Suppose $b = 0$ *and* $\sigma = 1$, *then* $P_t(x, dy) = p_t(x-y)dy$, *where,*

$$\forall t > 0, \forall u \in \mathbb{R}, \quad p_t(u) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right).$$

Then $(P_t)_{t\geq 0}$ *has the strong Feller property (even better:* $P_t\varphi$ *is smooth for all* $\varphi \in B_b(\mathbb{R})$ *!).*

Remark (PDE interpretation)

Let $u(t,x) := P_t\varphi(x)$. Then:

- In 1st example, u solves $\partial_t u = \partial_x u$, hyperbolic PDE.
- In 2nd example, u solves $\partial_t u = \frac{1}{2}\partial_x^2 u$, elliptic PDE.

What happens in general?

Theorem (Hörmander's criterion)

If σ is "sufficiently non-degenerate", then for all $t > 0$, $P_t\varphi$ is smooth for all $\varphi \in B_b(\mathbb{R})$. In particular, strong Feller property holds.

Original proof based on PDE techniques. Later (70s), P. Malliavin gave a proof using his calculus (integration by parts formulae).

The Bismut-Elworthy-Li formula

Problem: Hörmander's criterion difficult to adapt to other cases (singular coefficients, SPDEs). Not quantitative: no direct estimate of the continuity modulus of $P_t\varphi$ for $t > 0$.
Idea: Study the dependance of $X_t(x)$ w.r.t. x. Suppose $\sigma = 1$, for simplicity :

$$X_t(x) = x + \int_0^t b\left(X_s\left(x\right)\right) ds + B_t. \tag{2}$$

We assume $b : \mathbb{R} \to \mathbb{R}$ smooth and satisfying:

$$\begin{aligned} |b(x) - b(y)| &\leq C|x - y|, & x, y \in \mathbb{R} \\ b'(x) &\leq L, & x \in \mathbb{R} \end{aligned} \tag{3}$$

for some $C > 0$, $L \in \mathbb{R}$.

Setting $\eta_t(x) := \frac{dX_t(x)}{dx}$, we have:

$$\eta_t(x) = 1 + \int_0^t b'(X_s(x))\eta_s(x)ds.$$

This is an ODE with explicit solution:

$$\eta_t(x) := \exp\left(\int_0^t b'(X_s(x))\right).$$

Note that $0 \leq \eta_t(x) \leq e^{Lt}$.
Then, for $\varphi \in C_b^1(\mathbb{R})$, we have:

$$\frac{d}{dx}(P_t\varphi) = \frac{d}{dx}\mathbb{E}\left[\varphi(X_t(x))\right] = \mathbb{E}\left[\varphi'(X_t(x))\eta_t(x)\right].$$

Theorem (Bismut-Elworthy-Li formula)

For all $T > 0$ and $\varphi \in C_b(\mathbb{R})$, the function $P_T\varphi$ is differentiable and we have:

$$\frac{d}{dx} P_T\varphi(x) = \frac{1}{T}\mathbb{E}\left[\varphi(X_T(x)) \int_0^T \eta_s(x) dB_s\right].$$

Remark: This is an integration by parts formula with respect to the law of $(X_t)_{0\leq t\leq T}$ on the space of paths.

Corollary

The semi-group $(P_t)_{t\geq 0}$ is strongly Feller and, for all $T > 0$ and $\varphi \in B_b(\mathbb{R})$, one has:

$$|P_T\varphi(x) - P_T\varphi(y)| \leq e^{L} \frac{||\varphi||_\infty}{\sqrt{T \wedge 1}} |x - y|, \quad x, y \in \mathbb{R}.$$

An important remark: Above bound depends only on L, not on the Lipschitz constant C. Therefore Bismut-Elworthy-Li formula is very robust and well-adapted for proving strong Feller property for solutions of SPDEs.

Remark (History of the Bismut-Elworthy-Li formula)

A particular form of this formula had originally been derived by J.M. Bismut using Malliavin calculus, in the framework of the study of the logarithmic derivative of the fundamental solution of the heat equation on a compact manifold (1984). Ten years later (1994), K.D. Elworthy and X.-M. Li generalized this formula to a large class of diffusion processes on a manifold, and gave also variants of this formula to higher-order derivatives. Ever since, many interesting generalizations of BEL formula in various directions (SPDEs, degenerate diffusion coefficient, reflected SDEs, ...)

What if b' is not bounded above? Very important SPDEs are in this category!

In 2016, Tsatsoulis-Weber and Hairer-Mattingly have proved the strong Feller property for singular semilinear SPDEs like $P(\phi)_2$, KPZ, ϕ^4_3 and others. For this class of equations b is quite exotic and hardly dissipative. However, their approaches do not help us in a different class of problems.

Example

Consider evolution SPDEs of the form :

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + \frac{c}{u(t,x)^3} + W(t,x), \quad t \geq 0, \quad x \in (0,1),$$

with Dirichlet boundary condition on $[0,1]$. *Here* $c > 0$, W *is a space-time white noise on* $\mathbb{R} \times (0,1)$, *and we require* $u \geq 0$.

My advisor L. Zambotti proved well-posedness for these SPDEs as well as the strong Feller property for the corresponding semigroup on the state space:

$$E := \{h : [0,1] \to \mathbb{R}_+ \text{ continuous}, \quad h(0) = h(1) = 0\}.$$

He used the fact that $u \to \frac{c}{u^3}$ is decreasing on $\mathbb{R}_+$ (dissipativity of the drift).

Problem: what happens if $c < 0$, i.e. if the drift is non-dissipative? This is a very difficult problem ! A first step is to look at a family of one-dimensional diffusions that have similar difficulties, but are very nice to handle: the Bessel processes.

Bessel processes

Let $\delta > 0$ and $x \geq 0$. The δ - dimensional Bessel process started at x is the unique continuous, nonnegative process $(\rho_t(x))_{t\geq 0}$ solution to :

$$\begin{cases} d\rho_t = \frac{\delta-1}{2\rho_t}dt + dB_t & \text{when } \rho_t > 0, \\ \rho_0 = x, \end{cases} \tag{4}$$

such that the boundary point 0 is *instantaneously reflecting*.

Remark

When $\delta \in \mathbb{N}^*$, $(\rho_t)_{t\geq 0} \overset{(d)}{=} (||B_t||)_{t\geq 0}$, where B is a Brownian motion in $\mathbb{R}^\delta$.

SDE (4) is well-posed for $\delta \geq 1$, but when $\delta \in (0,1)$, drift coefficient is non-dissipative, and this SDE lies outside the framework of classical well-posedness results. Well-posedness of (4) follows from well-posedness of the SDE satisfied by $X_t := \rho_t^2$.

Remark

The process $(\rho_t(x))_{t\geq 0}$ is monotone non-decreasing in δ. In particular, setting $T_0(x) := \inf\{t > 0 : \rho_t(x) = 0\}$, we have:

- if $\delta \geq 2$, then $T_0(x) = +\infty$ a.s.,
- if $\delta \in [0, 2)$, then $T_0(x) < \infty$ a.s.

Let $P_t^\delta \varphi(x) := \mathbb{E}\left[\varphi(\rho_t(x))\right]$, $\varphi \in B_b(\mathbb{R}_+)$, $x \in \mathbb{R}_+$.

Question: strong Feller property for $\left(P_t^\delta\right)_{t\geq 0}$?

We use explicit formula: for all $T > 0$ and $x \geq 0$

$$P_T^\delta \varphi(x) = \int_0^{+\infty} p_T^\delta(x, y)\varphi(y)\, dy,$$

where $p_T^\delta(x, y)$ is an explicit, analytic function of $(x, y) \in \mathbb{R} \times \mathbb{R}_+^*$.

Theorem

For all $T > 0$ and $F \in B_b(\mathbb{R}_+)$, the function $x \to P_T^\delta F(x)$ is differentiable on $\mathbb{R}_+$, and:

$$\forall x \geq 0, \quad \frac{d}{dx} P_T^\delta F(x) = \frac{x}{T}\left(P_T^{\delta+2}F(x) - P_T^\delta F(x)\right). \tag{5}$$

In particular, $(P_t^\delta)_{t\geq 0}$ has the strong Feller property: for all $T > 0$, $R > 0$, and $x, y \in [0, R]$, we have:

$$|P_T^\delta F(x) - P_T^\delta F(y)| \leq \frac{2R||F||_\infty}{T}|y - x|.$$

Question Can we interpret equality (5) as a Bismut-Elworthy-Li formula ? The answer is yes when $\delta > 0$ and $x > 0$.

Proposition

Let $\delta > 0$, $t > 0$ and $x > 0$. Then, a.s., the function ρ_t is differentiable at x, and its derivative there is given by:

$$\left.\frac{d\rho_t(y)}{dy}\right|_{y=x} \overset{a.s.}{=} \eta_t(x) := \mathbf{1}_{t<T_0(x)} \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right). \qquad (6)$$

Remark

Note the indicator function $\mathbf{1}_{t<T_0(x)}$, which signals a coalescence of the Bessel flow when $\rho_t(x)$ hits the boundary 0. As a consequence, $\eta_t = 0$ for all $t \geq T_0(x)$!

Remark: Note that the smaller the dimension δ, the rougher the process $(\eta_t)_{t\geq 0}$.

- when $\delta > 1$, η is bounded and continuous on $\mathbb{R}_+$
- when $\delta = 1$, η is bounded but discontinuous at $T_0(x)$
- when $\delta \in (0,1)$, $\eta_t \to \infty$ as $t \nearrow T_0(x)$.

Question: what is the link between equality (5) and the BEL formula?
Let $(\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration on the space of paths $C(\mathbb{R}_+, \mathbb{R})$.

Proposition (cf. Pitman-Yor, 1981)

Let $\delta > 0$ and $x > 0$. Then, for all $t \geq 0$, the law ${P_x^{\delta+2}}_{|\mathcal{F}_t}$ is absolutely continuous w.r.t. the law $P^\delta_{x|\mathcal{F}_t}$, and the corresponding Radon-Nikodym derivative is given by:

$$\left.\frac{dP_x^{\delta+2}}{dP_x^\delta}\right|_{\mathcal{F}_t}(\rho) \overset{a.s.}{=} \mathbf{1}_{t<T_0(x)} \frac{\rho_t}{x} \exp\left(-\frac{\delta-1}{2}\int_0^t \frac{ds}{\rho_s^2}\right) = \frac{1}{x}\rho_t \eta_t.$$

Proposition

Let $\delta, x > 0$. The stochastic integral process $\left(\int_0^t \eta_s dB_s\right)_{t\geq 0}$ is well-defined as a local martingale and is indistinguishable from the **continuous** *martingale $(\rho_t\eta_t - x)_{t\geq 0}$.*

Proof of the last Proposition

Suppose $\delta \geq 1$. Then $\eta_t \in [0,1]$ for all $t \geq 0$, so $\int_0^t \eta_s dB_s$ is well-defined as an L^2 martingale. Hence by Itô's lemma we have:

$$\begin{aligned}
\rho_t \eta_t &= x + \int_0^t \eta_s \, d\rho_s + \int_0^t \rho_s \, d\eta_s \\
&= x + \int_0^{t \wedge T_0} \eta_s \, d\rho_s + \int_0^{t \wedge T_0} \rho_s \, d\eta_s \\
&= x + \int_0^{t \wedge T_0} \eta_s \left(\frac{\delta - 1}{2\rho_s} ds + dB_s \right) - \int_0^{t \wedge T_0} \rho_s \frac{\delta - 1}{2} \frac{\eta_s}{{\rho_s}^2} ds \\
&= x + \int_0^{t \wedge T_0} \eta_s dB_s \\
&= x + \int_0^t \eta_s dB_s,
\end{aligned}$$

so the claim follows (we used the fact that $\eta_s = 0$ for all $s \geq T_0$).

When $\delta \in (0,1)$, we cannot apply Itô's lemma directly anymore. Indeed, since η is not locally bounded, it is not clear that $\int_0^t \eta_s dB_s$ is well-defined as a local martingale. However, setting $p = \frac{(2-\delta)^2}{4(1-\delta)} \in (1,+\infty)$, we can prove that :

$$\mathbb{E}\left[\left(\int_0^t \eta_s^2 ds\right)^{p/2}\right] < \infty.$$

So $\int_0^t \eta_s dB_s$ is well-defined as a local martingale, and is in L^p. Applying Itô's Lemma to the stopped process $(\rho_{t\wedge T_\epsilon}\eta_{t\wedge T_\epsilon})_{t\geq 0}$ where T_ϵ is the first hitting time of $\epsilon > 0$, one has:

$$\int_0^{t\wedge T_\epsilon} \eta_s dB_s = \rho_{t\wedge T_\epsilon}\eta_{t\wedge T_\epsilon} - x.$$

Since the process $(\rho_t\eta_t)_{t\geq 0}$ is continuous (not trivial...), using the BDG inequality, we can send $\epsilon \to 0$ to obtain the claimed equality.

Theorem (Bismut-Elworthy-Li formula)

Let $\delta > 0$. Then, for all $T > 0$ and all $F \in B_b(\mathbb{R}_+)$, the function $x \to P_T^\delta F(x)$ is differentiable on $\mathbb{R}_+$, and for all $x > 0$:

$$\frac{d}{dx} P_T^\delta F(x) = \frac{1}{T} \, \mathbb{E}\left[F(\rho_t(x)) \left(\int_0^T \eta_s(x) dB_s \right) \right].$$

In particular, for all $\delta \geq 2(\sqrt{2}-1)$, for all $R > 0$, there exists $C > 0$ such that, for all $x, y \in [0, R]$, we have:

$$|P_T^\delta F(x) - P_T^\delta F(y)| \leq \frac{C||F||_\infty}{T^{\alpha(\delta)}} \, |y - x| \tag{7}$$

where the exponent $\alpha(\delta) \in [\frac{1}{2}, 1)$ is given by:

$$\alpha(\delta) := \begin{cases} \frac{1}{2} + \frac{1-\delta}{2-\delta} & \textit{if } \delta \in [2(\sqrt{2}-1), 1], \\ 1/2 & \textit{if } \delta \geq 1. \end{cases}$$

Note that $\frac{1}{2} < \alpha(\delta) < 1$. Hence this strong Feller bound is an improvement of the a priori bound obtained previously (in $1/T$), although it is a little worse than in the dissipative case (bound in $1/\sqrt{T}$).

Remark

The number $2(\sqrt{2}-1) \approx 0.83$ is the minimal value of δ for which the martingale $\left(\int_0^t \eta_s dB_s\right)_{t \geq 0}$ is in L^2; for $\delta < 2(\sqrt{2}-1)$, this martingale is only in L^p for some $1 < p < 2$. Since the calculations performed to obtain the above strong Feller bound are based on a Jensen inequality which requires $p \geq 2$, we must make the awkward assumption $\delta \geq 2(\sqrt{2}-1)$. However it seems reasonable to conjecture that the above bound holds for all $\delta \in (0,1)$ (note that $\alpha(\delta) \underset{\delta \to 0}{\nearrow} 1$).

Conclusion

Our results, in particular the Bismut-Elworthy-Li formula, rely on very specific properties of Bessel processes (absolute continuity results, etc). However they prove that the strong Feller property and even the Bismut-Elworthy-Li formula can hold for radically non-dissipative systems. Hence we suspect that similar properties might hold for solutions to more general (S)PDEs with non-dissipative drift.
A recent result goes in that direction: consider the solution $(X_t)_{t\geq 0}$ to a reflected SDE in $\mathbb{R}_+$

$$\begin{cases} dX_t = b(X_t)dt + dB_t + d\ell_t \\ X_t \geq 0, \ d\ell_t \geq 0, \ X_t d\ell_t = 0, \end{cases}$$

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth.

Proposition

Assume that the function $\mathbb{R}_+ \to \mathbb{R}, x \to xb(x)$ *is non-increasing. Then for all* $\varphi \in B_b(\mathbb{R}_+)$*, all* $T > 0$ *and all* $x, y \in [0, R]$*:*

$$|P_T\varphi(x) - P_T\varphi(y)| \leq 2R\frac{||\varphi||_\infty}{T}|x - y|.$$

Example

Assume that $b(x) = cx^\alpha$ *for* $c, \alpha \in \mathbb{R}$*. Then* $(X_t)_{t\geq 0}$ *satisfies the assumption of the proposition iff one of the following conditions holds:*

- *either* $\alpha < -1$ *and* $c \geq 0$*,*
- *or* $\alpha > -1$ *and* $c \leq 0$*,*
- *or* $\alpha = -1$*.*

The case $\alpha = -1$ *corresponds to the Bessel processes (of all dimensions).*

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