# Identification and isotropy characterization of deformed random fields through excursion sets

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# The model of deformed random fields.

- Let $X : \mathbb{R}^2 \to \mathbb{R}$ be a **stationary and isotropic random field**: for any translation $\tau$, for any rotation $\rho$ in $\mathbb{R}^2$,

$$X \circ \tau \overset{\text{law}}{=} X \quad \text{and} \quad X \circ \rho \overset{\text{law}}{=} X.$$

  We write $C(t) = \mathrm{Cov}(X(t), X(0))$ its covariance function.
  We call $X$ the **underlying field**.

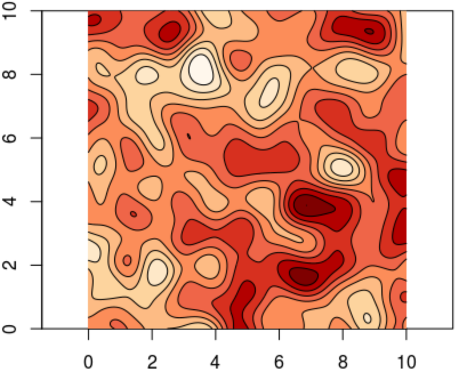
- let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective, bicontinuous, deterministic application satisfying $\theta(0) = 0$, which we will call a **deformation**.

$X_\theta = X \circ \theta : \mathbb{R}^2 \to \mathbb{R}$ is the **deformed random field** constructed with the underlying field $X$ and the deformation $\theta$.

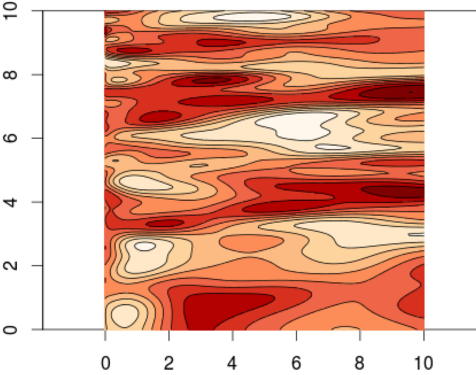
Two types of questions :

- Invariance properties of the deformed field
- Inverse problem: identification of $\theta$ thanks to (partial) observations of $X_\theta$.

**First observation:** the invariance properties are not preserved in general.




Level sets of a realization of a Gaussian stationary and isotropic random field $X$ with Gaussian covariance $C(x) = \exp(-\|x\|^2)$.




Level sets of a realization of $X_\theta$ constructed with $\theta : (s,t) \mapsto (s^{0.6}, t^{1.4})$ and with the underlying field $X$.

**Question**

Which are the deformations that preserve stationarity and isotropy ?

# References

- Spatial statistics (Sampson and Guttorp, 1992).
- Image analysis : "shape from texture" issue (Clerc-Mallat, 2002).
- Numerous domains of application in physics:
  for instance, used in cosmology for the modelization of the CMB and mass reconstruction in the universe.
- Also studied by Cabaña, 1987,Perrin-Meiring, 1999; Perrin-Senoussi, 2000, etc..

# Cases of isotropy (in law)

## Our assumptions

The **underlying field** $X$ must satisfy the following assumptions :

$$\textbf{(H)} \begin{cases} X \text{ is stationary and isotropic,} \\ X \text{ is centered and admits a second moment.} \end{cases}$$

The **deformation** $\theta$ belongs to the set

$$\begin{aligned} \mathcal{D}^0(\mathbb{R}^2) = \{\theta : \mathbb{R}^2 \to \mathbb{R}^2 \,/\, \theta & \text{ is continous and bijective,} \\ & \text{with a continuous inverse,} \\ & \text{such that } \theta(0) = 0\} \end{aligned}$$

# Cases of isotropy

**Problem**

Which are the deformations $\theta$ such that **for any underlying random field X**, $X_\theta$ is isotropic ?

- **Example :** elements of SO(2) : rotations of $\mathbb{R}^2$.
- **Another problem** : Which are the deformations $\theta$ such that for a fixed underlying random field $X$, $X_\theta$ is isotropic ?
- **For the proof.**
  - Invariance of the covariance function of $X_\theta$ under rotations :
$$\forall \rho \in SO(2), \forall (x,y) \in (\mathbb{R}^2)^2,$$
$$\mathrm{Cov}(X_\theta(\rho(x)), X_\theta(\rho(y))) = \mathrm{Cov}(X_\theta(x), X_\theta(y))$$
$$C(\theta(\rho(x)) - \theta(\rho(y))) = C(\theta(x) - \theta(y))$$
  - Chose the covariance function $C(x) = \exp(-\|x\|^2)$ to obtain
$$\forall \rho \in SO(2),\ \forall (x,y) \in (\mathbb{R}^2)^2, \quad \|\theta(\rho(x)) - \theta(\rho(y))\| = \|\theta(x) - \theta(y)\|.$$
  - Polar representation of $\theta$.

# Cases of isotropy

## Answer to the problem

**Spiral deformations** are the deformations preserving isotropy for any underlying field $X$.

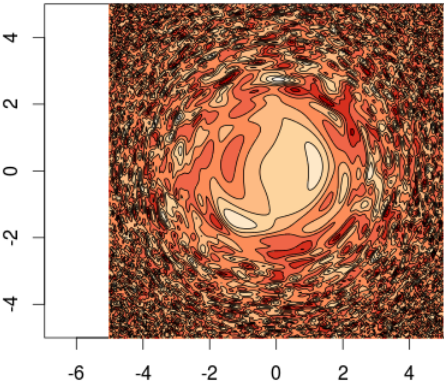
**Notations** : $\hat{\theta}$ polar representation of $\theta$ :
$\hat{\theta} : (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z} \to (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z} \quad (r,\varphi) \mapsto (\hat{\theta}_1(r,\varphi), \hat{\theta}_2(r,\varphi))$.

## Definition

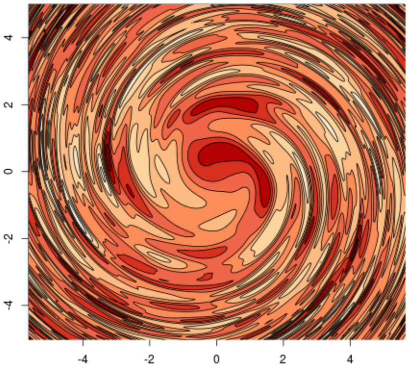
*A deformation* $\theta \in \mathcal{D}^0(\mathbb{R}^2)$ *is a* **spiral deformation** *if there exist* $f : (0,+\infty) \to (0,+\infty)$ *strictly increasing and surjective,* $g : (0,+\infty) \to \mathbb{Z}/2\pi\mathbb{Z}$ *and* $\varepsilon \in \{\pm 1\}$ *such that* $\theta$ *satisfies*

$$\forall (r,\varphi) \in (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z}, \quad \hat{\theta}(r,\varphi) = (f(r), g(r) + \varepsilon\varphi).$$

# Simulations of fields deformed with spiral deformations




Level sets of a realization of $X_\theta$, with a deformation $\theta : x \mapsto \|x\| \, x$ and $X$ Gaussian with Gaussian covariance.

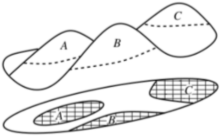


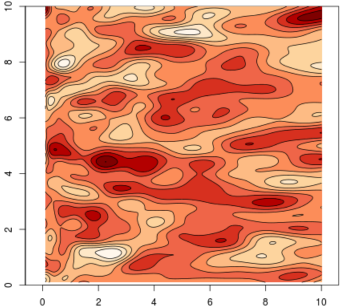

Level sets of a realization of $X_\theta$, with $\theta$ a deformation with polar representation $\hat{\theta} : (r, \varphi) \mapsto (\sqrt{r}, r + \varphi)$ and $X$ Gaussian with Gaussian covariance.

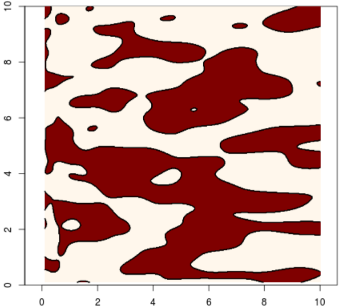
# Excursion sets

- Let $u \in \mathbb{R}$ be a fixed level,
- let $T$ be a rectangle or a segment in $\mathbb{R}^2$,
- let $A_u(X_\theta, T)$ be the **excursion set** of $X_\theta$ restricted to $T$ above level $u$:

$$\boxed{A_u(X_\theta, T) = \{t \in T \,/\, X_\theta(t) \geq u\}}$$




Level sets and excursion sets of a realization of $X_\theta$, with $\theta : (s,t) \mapsto (s^{0.6}, t)$ defined on $(0, +\infty)^2$ and $X$ Gaussian with Gaussian covariance.

# Euler characteristic $\chi$ of excursion sets

**Euler characteristic:** integer-valued and additive functional defined on a large class of compact sets.

**Heuristic definition for a compact set $G \subset \mathbb{R}^2$ of dimension 1 or 2**

- $d = 1$, $\chi(G) = \#$(disjoint components in G);
- $d = 2$, $\chi(G) = \#$(connected components in G) $-$ $\#$(holes in G).

The Euler characteristic is a homotopy invariant, hence

$$A_u(X_\theta, T) = \theta^{-1}(A_u(X, \theta(T)) \quad \Rightarrow \quad \boxed{\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T)))}.$$

and we can use an expectation formula proven for stationary and isotropic random fields in Adler-Taylor, 2007.

# Additional assumptions

$$
\textbf{(H')} \begin{cases} \textbf{X is Gaussian}, \\ X \text{ is stationary and isotropic}, \\ \textbf{X is almost surely of class } \mathcal{C}^{\mathbf{2}}, \\ X \text{ is centered, } C(0) = 1 \text{ and } C''(0) = -I_2, \\ \textbf{a non-degeneracy assumption on } \mathbf{X(t)}, \textbf{ for every } \mathbf{t} \in \mathbb{R}^{\mathbf{2}}. \end{cases}
$$

The deformation $\theta$ belongs to the set

$$
\begin{aligned}
\mathcal{D}^2(\mathbb{R}^2) = \{\theta : \mathbb{R}^2 \to \mathbb{R}^2 \,/\, \theta \ \textbf{of class } \mathcal{C}^{\mathbf{2}} \text{ and bijective,} \\
\text{with an inverse } \textbf{of class } \mathcal{C}^{\mathbf{2}}, \\
\text{such that } \theta(0) = 0\}
\end{aligned}
$$

# Formulas for the expectation of $\mathbb{E}[\chi(A_u(X_\theta, T))]$

• If $T$ is a segment in $\mathbb{R}^2$, writing $|\theta(T)|_1$ the one-dimensional Hausdorff measure of $\theta(T)$,

$$\boxed{\mathbb{E}[\chi(A_u(X_\theta, T))] = e^{-u^2/2}\frac{|\theta(T)|_1}{2\pi} + \Psi(u)},$$

where $\Psi(u) = \mathbb{P}(Y > u)$ for $Y \sim \mathcal{N}(0,1)$.

• If $T \subset \mathbb{R}^2$ is a rectangle, writing $|\theta(T)|_2$ the two-dimensional Hausdorff measure of $\theta(T)$,

$$\boxed{\mathbb{E}[\chi(A_u(X_\theta, T))] = e^{-u^2/2}\left(u\frac{|\theta(T)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T)|_1}{4\pi}\right) + \Psi(u)},$$

where $\partial G$ is the frontier of $G$.

Writing $\theta = (\theta_1, \theta_2)$ the coordinate functions of $\theta$, let $J_\theta(s,t)$ be the **Jacobian matrix** of $\theta$ at point $(s,t) \in \mathbb{R}^2$ :

$$J_\theta(s,t) = \begin{pmatrix} \frac{\partial \theta_1}{\partial s}(s,t) & \frac{\partial \theta_1}{\partial t}(s,t) \\ \frac{\partial \theta_2}{\partial s}(s,t) & \frac{\partial \theta_2}{\partial t}(s,t) \end{pmatrix} = \begin{pmatrix} J_\theta^1(s,t) & J_\theta^2(s,t) \end{pmatrix}.$$

Note that the Jacobian determinant is either positive on $\mathbb{R}^2$ or negative on $\mathbb{R}^2$.

- $|\theta([0,s] \times [0,t])|_2 = \int_0^s \int_0^t |\det(J_\theta(x,y))| \, dx \, dy$
- $|\theta([0,s] \times \{t\})|_1 = \int_0^s \sqrt{\partial_x \theta_1(x,t)^2 + \partial_x \theta_2(x,t)^2} \, dx = \int_0^s \|J_\theta^1(x,t)\| \, dx$
- $|\theta(\{s\} \times [0,t])|_1 = \int_0^t \sqrt{\partial_y \theta_1(s,y)^2 + \partial_y \theta_2(s,y)^2} \, dy = \int_0^t \|J_\theta^2(s,y)\| \, dy.$

**Consequence : general idea**

Condition / information on $\mathbb{E}[\chi(A_u(X, \theta(T)))]$ (T rectangle or segment) implies condition / information on the Jacobian matrix of $\theta$, hence on $\theta$.

# A weak notion of isotropy linked to excursion sets

Let $X$ be an underlying field satisfying **(H')**.

**Definition ($\chi$-isotropic deformation)**

*A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is $\chi$-isotropic if for any rectangle $T$ in $\mathbb{R}^2$, for any $u \in \mathbb{R}$ and for any $\rho \in SO(2)$,*

$$\mathbb{E}[\chi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\chi(A_u(X_\theta, T)].$$

- First observation : $\theta$ spiral deformation $\Rightarrow$ $\theta$ $\chi$-isotropic deformation
- Therefore, if $\theta$ $\chi$-isotropic, $X_\theta$ can be considered as **weakly isotropic**.
- Definition depending on the underlying field $X$.

**Aim :** Prove that

$\theta$ $\chi$-isotropic deformation $\quad\Rightarrow\quad$ $\theta$ spiral deformation.

# First characterization

- The $\chi$-isotropic condition is also true for $T$ segment.
- Formulas for $\mathbb{E}[\chi(A_u(X_\theta, T)]$ involve $J_\theta$,
  formulas for $\mathbb{E}[\chi(A_u(X_\theta, \rho(T))]$ involve $J_{\theta\circ\rho}$.

**Lemma 1**

A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is $\chi$-isotropic if and only if for any $\rho \in SO(2)$, for any $x \in \mathbb{R}^2$,

$$\begin{cases} (i) & \forall k \in \{1,2\},\ \|J^k_{\theta\circ\rho}(x)\| = \|J^k_\theta(x)\|, \\ (ii) & \det(J_{\theta\circ\rho}(x)) = \det(J_\theta(x)). \end{cases}$$

# Second characterization and conclusion of the proof

A translation of the first lemma in polar coordinates brings:

**Lemma 2**

A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is a $\chi$-isotropic deformation if and only if functions

$$\begin{cases} (r,\varphi) \mapsto (\partial_r \hat{\theta}_1(r,\varphi))^2 + (\hat{\theta}_1(r,\varphi)\,\partial_r \hat{\theta}_2(r,\varphi))^2 \\ (r,\varphi) \mapsto (\partial_\varphi \hat{\theta}_1(r,\varphi))^2 + (\hat{\theta}_1(r,\varphi)\,\partial_\varphi \hat{\theta}_2(r,\varphi))^2 \\ (r,\varphi) \mapsto \hat{\theta}_1(r,\varphi)\,\det(J_{\hat{\theta}}(r,\varphi)) \end{cases}$$

are radial, i.e. if they do not depend on $\varphi$.

This differential system is solved in Briant, Fournier (2017, submitted) and **the set of solutions is exactly the set of spiral deformations**.

# Chain of equalities

We write

- $\mathcal{S}$ the set of spiral deformations in $\mathcal{D}^2(\mathbb{R}^2)$,
- $\mathcal{I}$ the set of deformations $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ such that **for any underlying field** $X$ satisfying $(\mathbf{H}')$, $X_\theta$ is isotropic,
- for a **fixed** underlying field $X$ satisfying $(\mathbf{H}')$,

$$\mathcal{I}(X) = \{\theta \in \mathcal{D}^2(\mathbb{R}^2) \text{ such that } X_\theta \text{ is isotropic}\}.$$

- $\mathcal{X}$ the set of $\chi$-isotropic deformations.

**Corollary**

*Let $X$ be a stationary and isotropic random field satisfying $(\mathbf{H}')$. Then $\mathcal{S} = \mathcal{I} = \mathcal{I}(X) = \mathcal{X}$.*

**Conclusion :** A weak notion of isotropy based on excursion sets coincides with isotropy in law.

# Identification of the deformation

Different methods exist, but most of them require to know the deformed field on a whole window (see Guyon-Perrin (2000), Clerc-Mallat (2003), Anderes-Stein (2008), Anderes-Chatterjee (2009), Anderes-Guiness (2016), etc.).

## Framework

We assume that **the deformation $\theta$ is unknown**.

We only have at our disposal **sparse data**: the observations of **one excursion set of $\mathbf{X}_\theta$** restricted to a certain window **above a fixed level $\mathbf{u} \neq \mathbf{0}$**.

(Additional assumptions on $\theta$)

## Claim

Let us assume that, for one level $u \neq 0$, we know $\mathbb{E}[\chi(A_u(X_\theta, T))]$ for every rectangle or segment $T$ in a fixed window $W$.

Then at each point $x \in W$, we may compute $\|J_\theta^1(x)\|$, $\|J_\theta^1(x)\|$ and $\det(J_\theta(x))$.

Consequently, the complex dilatation at point $x$ is determined, up to complex conjugation.

**Thanks for your attention !**

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