Constructing the fractional Brownian motion

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11 Mai 2017
Les probabilités de demain
IHÉS

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Outline

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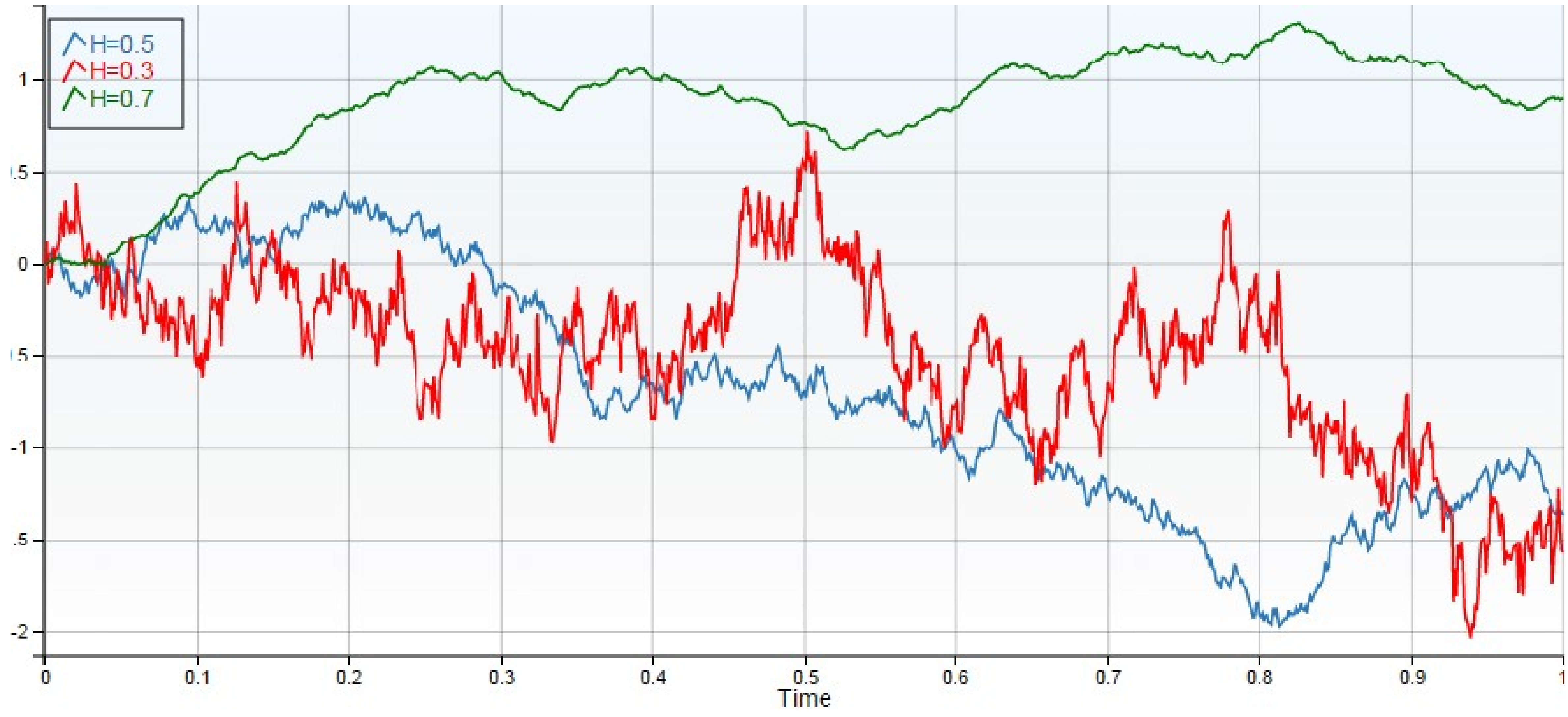
- Constructing the fractional Brownian motion
- Conclusion

Introduction

Definition

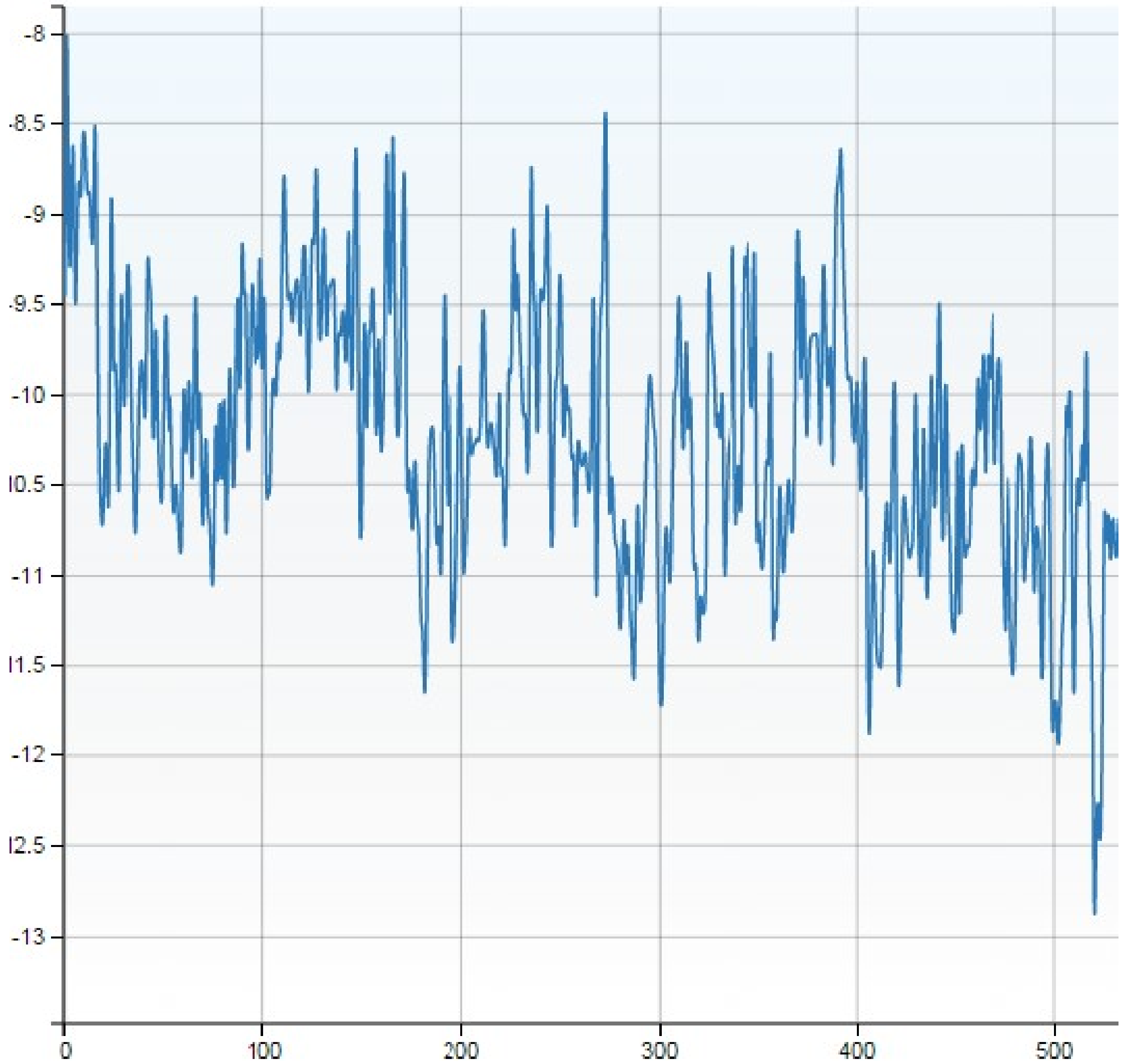
- fBm with Hurst exponent H$\in (0,1)$, is a centered Gaussian process $(X_t)_{t\geq 0}$ such that:

$$\forall s,t \geq 0, \quad \mathbb{E}X_sX_t = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

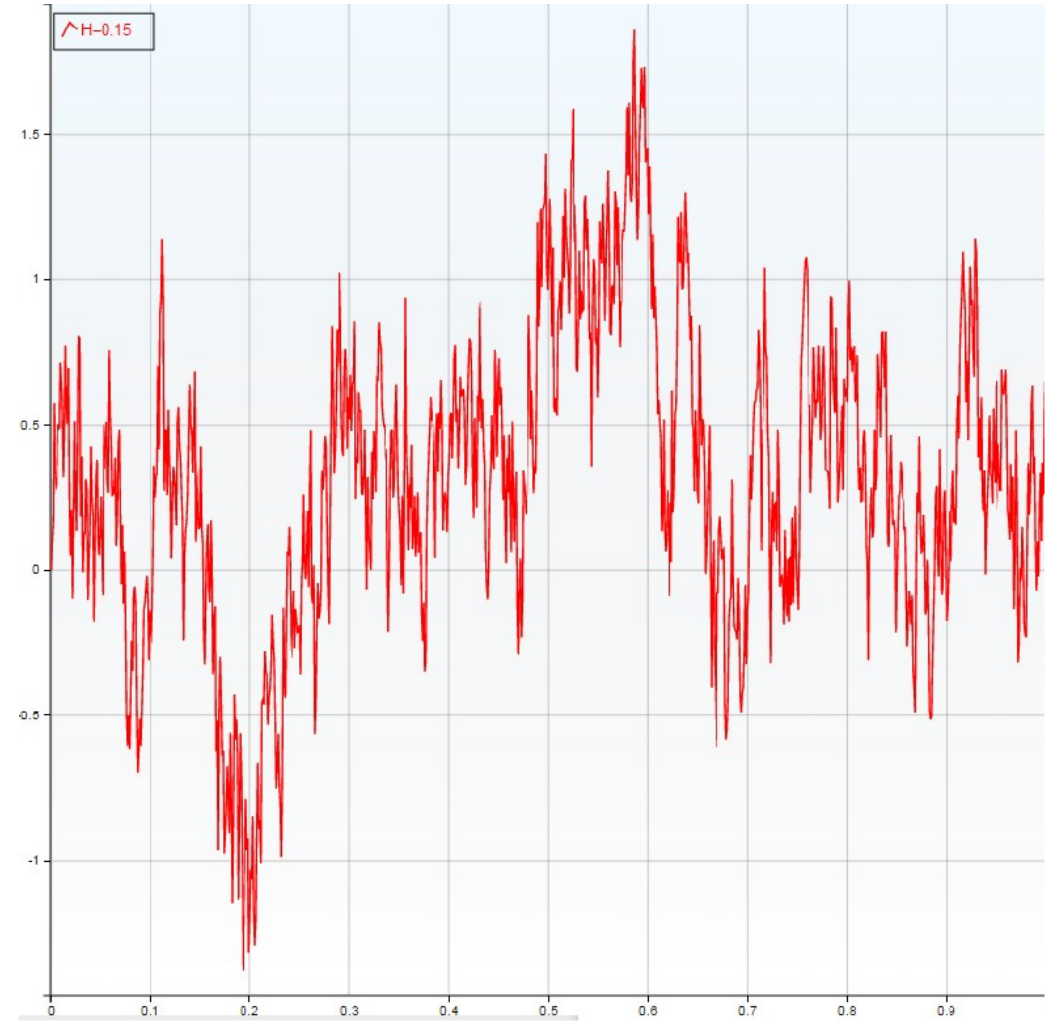


Application of fBm

- fBm is not used to model prices because it provides an arbitrage.
- fBm can be used to drive the dynamics of the log variance (Cf. Gatheral)



Real data



Simulated data

Simulation of fBm

One of the technical difficulties of fbm is simulation.

- Cholesky decomposition. (N^3 decomposition)
- Approximate circulant method.
- Series expansion. (Karhunen-Loève decomposition)

Series expansion for continuous stochastic processes

Definition

Let X be a closed set in $\mathbb{R}^n$, μ a strictly positive Borel measure on X, K a continuous function on $X \times X$. Define the operator T_K such that:

$$T_K : L^2(X) \to L^2(X),$$

$$\forall g \in L^2(X), \forall x \in X, \quad T_K(g)(x) = \int_X K(x,y)g(y)d\mu(y).$$

We will assume that K satisfies:

- for all square integrable functions g, we have

$$\langle T_K(g), g \rangle \geq 0.$$

- $$\int_X K(x,x)d\mu(x) < +\infty$$

Mercer's theorem

Theorem:

There is an orthonormal set (ψ_i) of $L^2(X)$ consisting of eigenfunctions of T_K such that corresponding sequence of eigenvalues (λ_i) is nonnegative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on X and K has the representation:

$$K(x, y) = \sum_{k=1}^{\infty} \lambda_k \psi_k(x) \psi_k(y).$$

The series converges absolutely for $(x, y) \in X \times X$ and uniformly on each compact subset of X.

- Application to the Kernel Trick.

Karhunen-Loève theorem

Theorem:

Let (X_t) be a centered Gaussian process over [0,1], with continuous covariance function, then X admits the following representation:

$$\forall t \in [0,1], \quad X_t = \sum_{k=1}^{\infty} Z_k \psi_k(t),$$

where $(\psi_k)_{k\geq 1}$ is an orthonormal basis of $L^2([0,1])$, and $(Z_k)_{k\geq 1}$ are independent centered Gaussian random variables.

- A general version of this theorem exists.

Sketch of the proof

The K-L decomposition is based on Mercer's theorem:

- $$\forall s, t \in [0,1], \quad K(s,t) = \sum_{k=1}^{\infty} \lambda_k \psi_k(s) \psi_k(t).$$
- $\lambda_k \geq 0.$

It comes out that X admits the following representation:

$$\forall t \in [0,1], \quad X_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k \psi_k(t).$$

Constructing the fractional Brownian motion

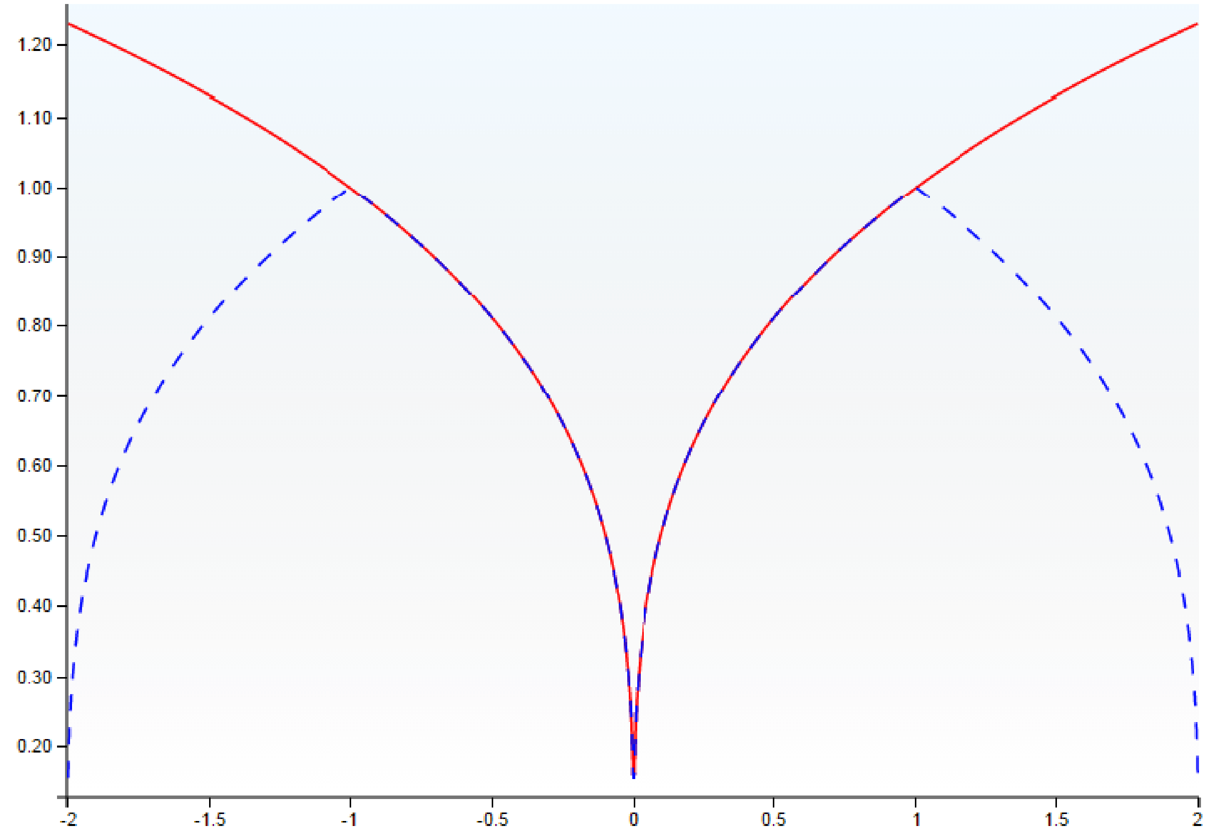
Constructing fBm

- The extension of the auto-covariance function into a 2T- periodic function gives:

$$\forall t \in [-1,1], \quad |t|^{2H} = \sum_{k=1}^{\infty} c_k(\cos k\pi t - 1),$$

$$\text{where} \quad c_k = \int_0^1 t^{2H} \cos k\pi t dt$$

$c_k = \underset{k\to\infty}{\mathcal{O}}\left(k^{-1-2H}\right)$

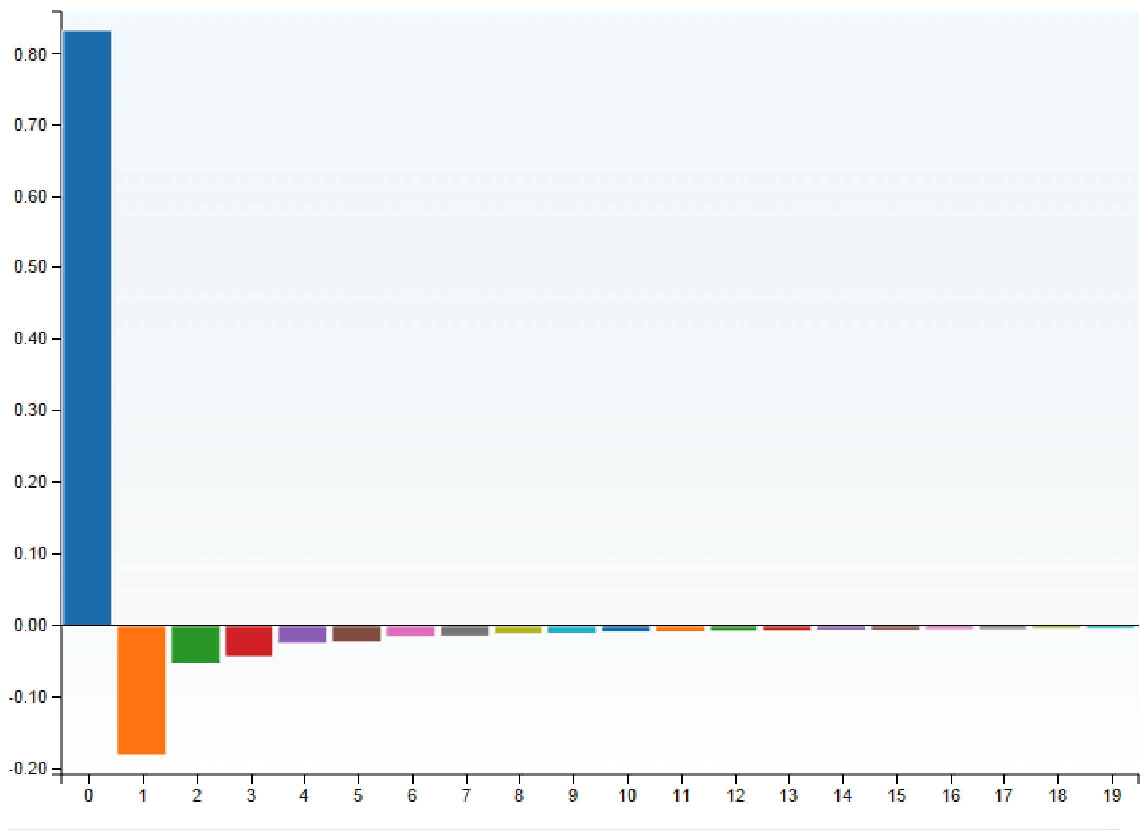


Constructing fBm

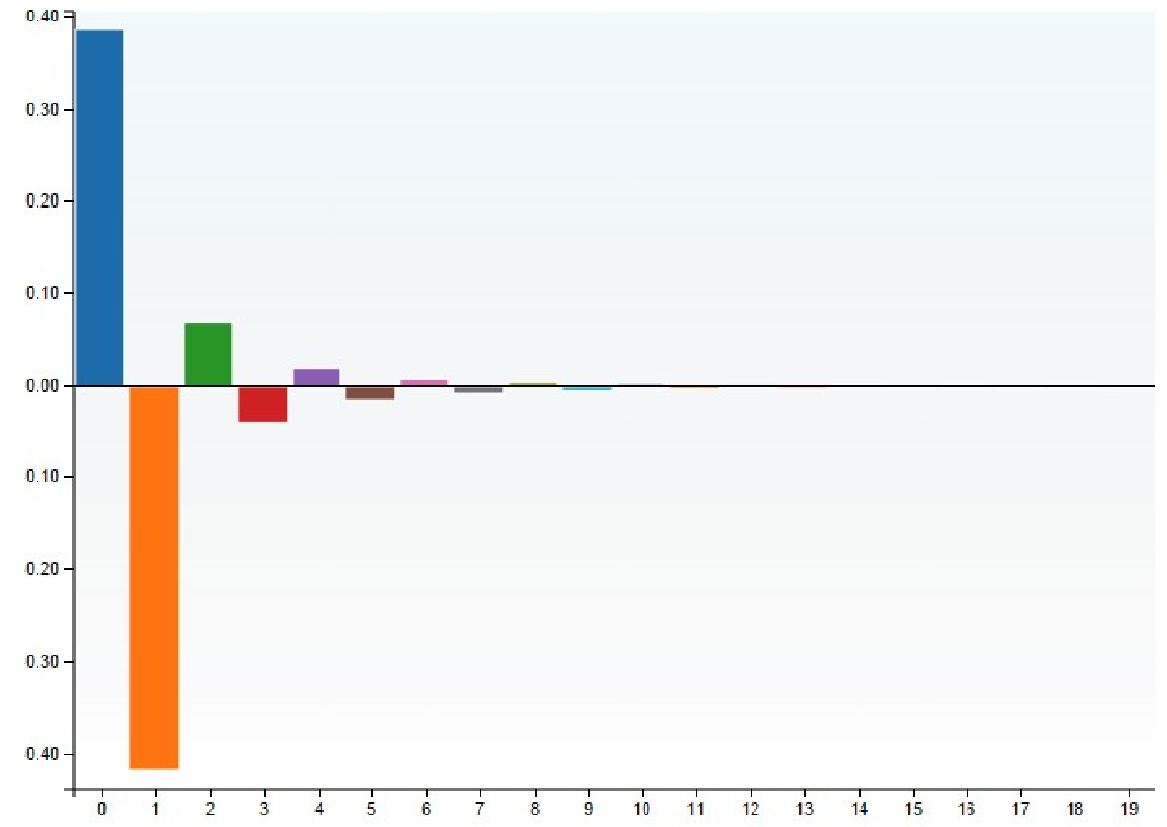
- We get a decomposition of the covariance function:

$$
\begin{aligned}
K(t,s) &= \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right) \\
&= \sum_{k=1}^{\infty} \frac{c_k}{2} (\cos k\pi t + \cos k\pi s - \cos k\pi (t-s) - 1) \\
&= \sum_{k=1}^{\infty} \frac{-c_k}{2} (1 - \cos k\pi t)(1 - \cos k\pi s) \\
&\quad + \sum_{k=1}^{\infty} \frac{-c_k}{2} \sin k\pi t \sin k\pi s.
\end{aligned}
$$

Fourier spectrum of the fractional Kernel



$H < 0.5$

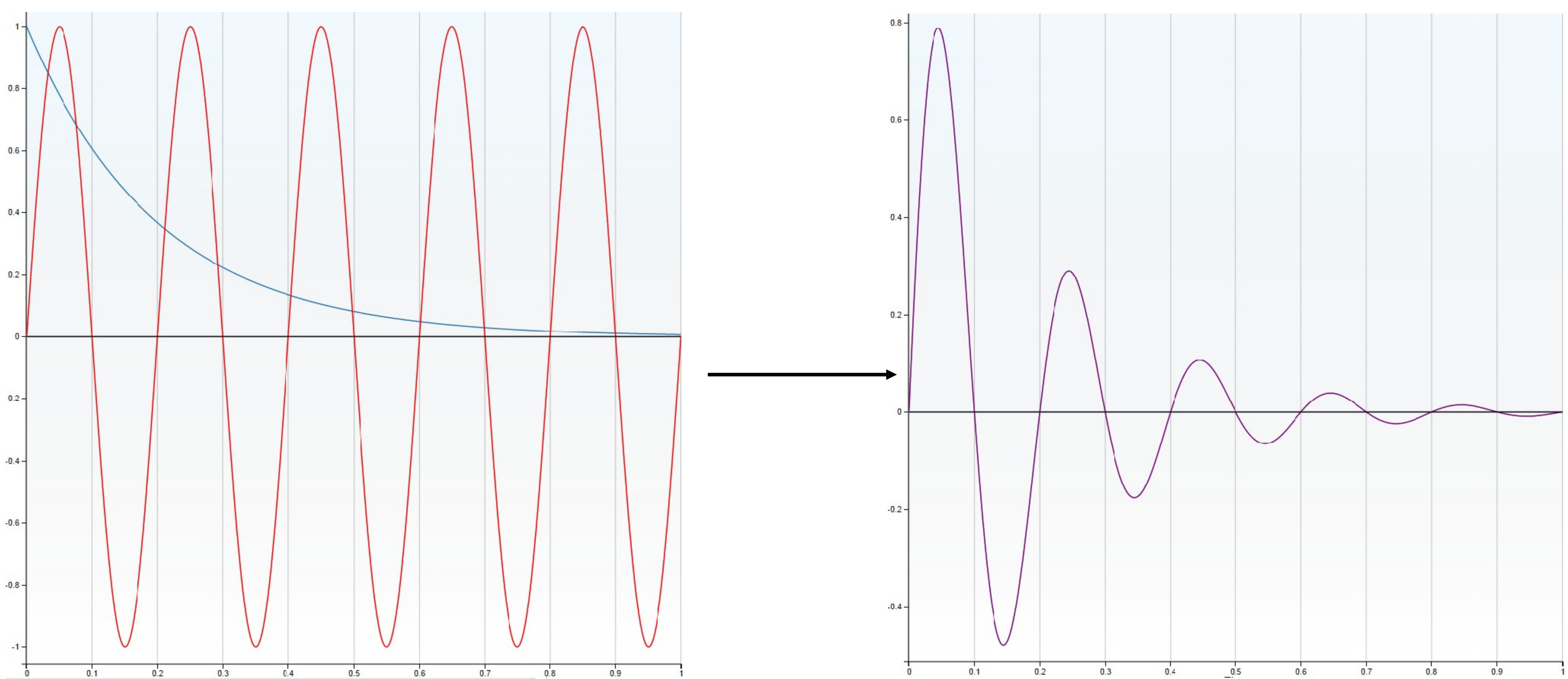


$H > 0.5$

Constructing fBm

- The extended covariance function is not necessarily a covariance function.

$$\forall k \geq 1, \quad -c_k = -\int_0^1 t^{2H} \cos k\pi t dt = 2H \int_0^1 t^{2H-1} \sin k\pi t dt.$$



$H < 0.5$

The series expansion

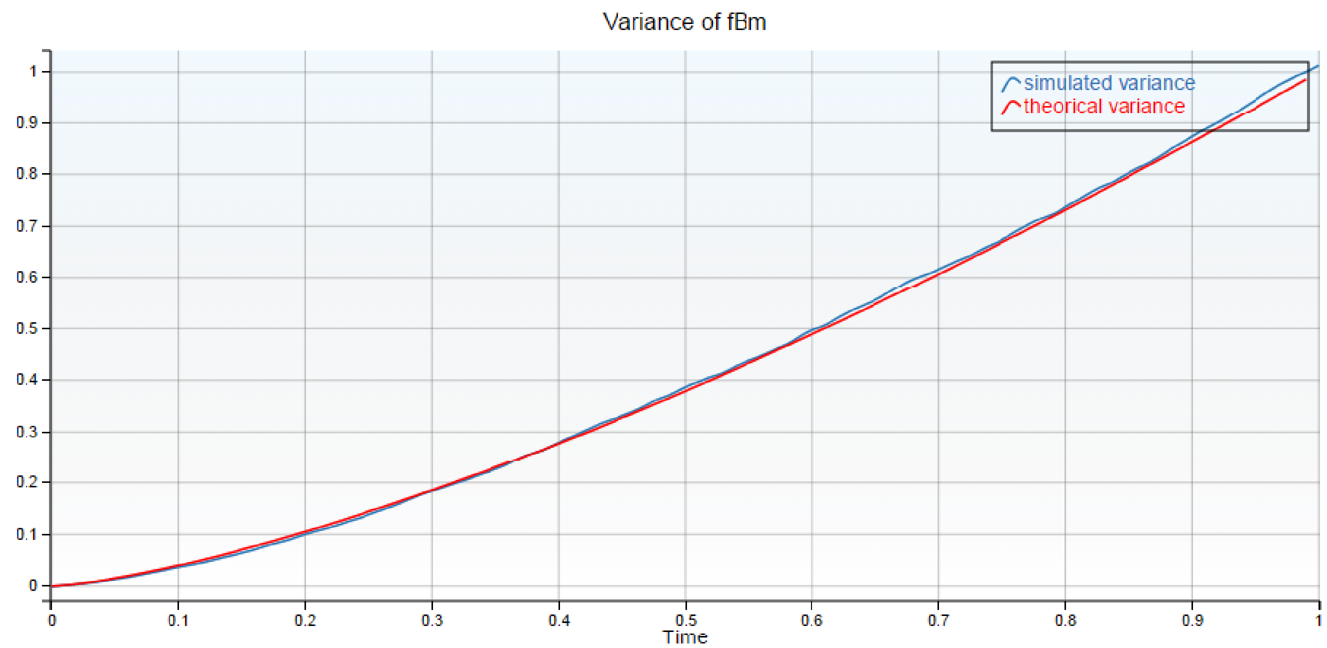
- X admits the following series expansion for $H < \frac{1}{2}$:

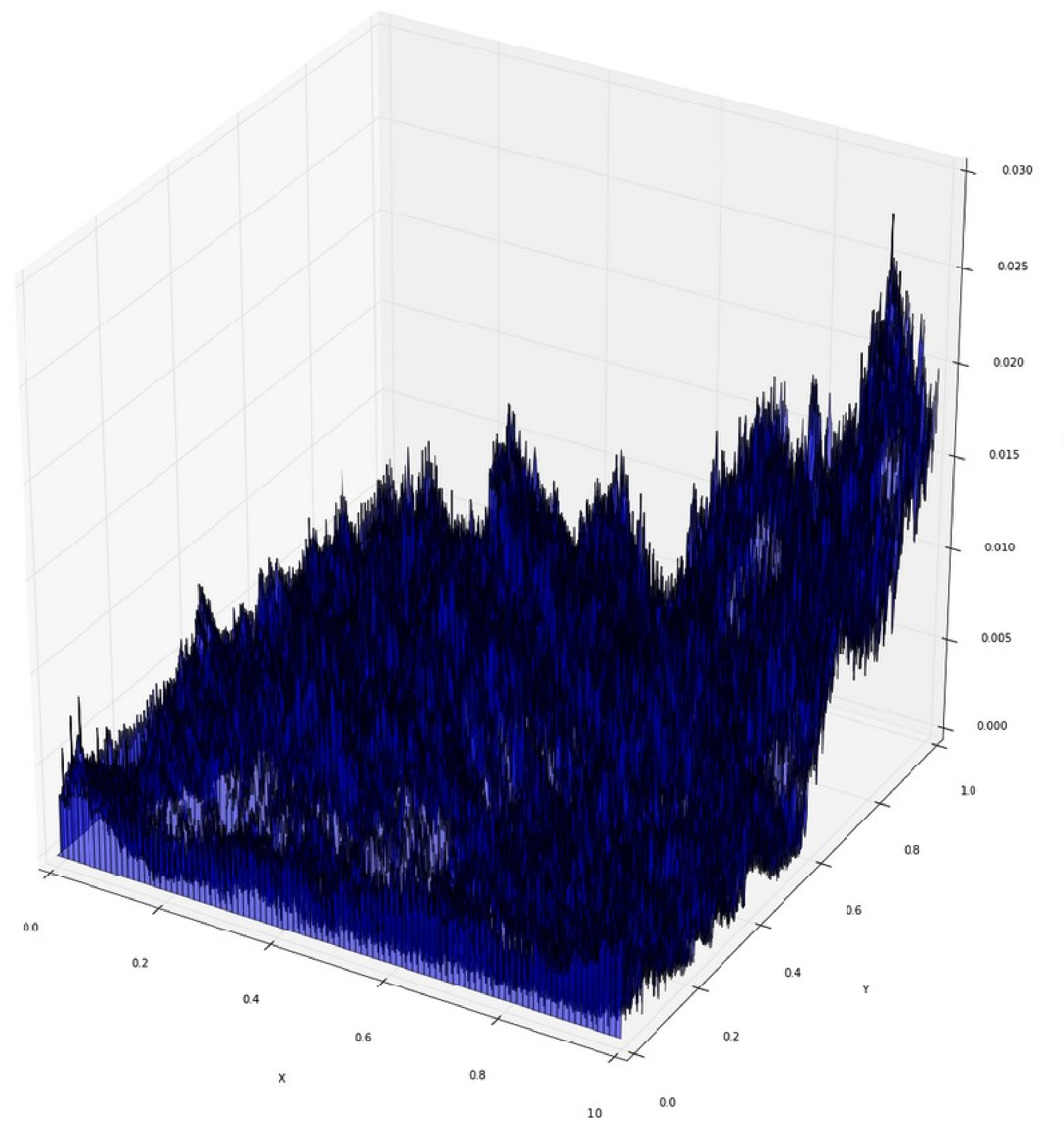
$$\forall t \in [0,1], \quad X_t = \sum_{k=1}^{\infty} \sqrt{\frac{-c_k}{2}} \left(Z_k \left(1 - \cos k\pi t\right) + Z_{-k} \sin k\pi t \right).$$

- This series converges uniformly, almost surely, and is rate-optimal:

$$\mathbb{E} \sup_{t\in[0,T]} \left| X_t - X_t^N \right| \underset{N\to\infty}{\sim} AN^{-H} \sqrt{\log(N)}, \; A > 0.$$

Simulation of fBm





$H > 0.5$

$H < 0.5$

Previous expansions

In [1], Dzhaparidze and van Zanten discovered the following series expansion for fBm

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos y_n t}{y_n} Y_n, \quad t \in [0,1]$$

where $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are i.i.d centered Gaussian random variables, $(x_n)_{n\geq 1}$ the positive roots of the Bessel function J_{-H}, and $(y_n)_{n>1}$ the positive roots of the Bessel function J_{1-H}. The variance of the Gaussian variables is given by: $VarX_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n)$, $VarY_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n)$, where $c_H^2 = \pi^{-1}\Gamma(1+2H)\sin \pi H$. In their paper, they prove that this expansion is rate-optimal in the following sense:

In [2], Igloi gives another rate-optimal series expansion for fBm in the case H>1/2 which is similar to our representation in that it is based on the same frequencies. This expansion is of the form

$$B_t = a_0 t X_0 + \sum_{k=1}^{\infty} a_k \Big(\sin(k\pi t) X_k + (1 - \cos(k\pi t)) X_{-k} \Big), \quad t \in [0,1]$$

where

$$a_0 = \sqrt{\frac{\Gamma(2-2H)}{B(H-\frac{1}{2}, \frac{3}{2}-H)(2H-1)}},$$

$$\forall k \in \mathbb{N}^*, \quad a_k = \sqrt{\frac{\Gamma(2-2H)}{B(H-\frac{1}{2}, \frac{3}{2}-H)(2H-1)} 2\Re(i \exp^{-i\pi H} \gamma(2H-1, ik\pi))(k\pi)^{-H-\frac{1}{2}}},$$

and $(X_k)_{k\in\mathbb{Z}}$ are i.i.d standard Gaussian random variables.

Conclusion

Conclusion

- The case $H > 0.5$ can be handled by a slight modification of the covariance function.
- Generalization of the decomposition to a large class of Gaussian processes with stationarity.
- Series expansions are interesting to simulate non Markovian Gaussian processes, and control precisely the approximation error.
- Potentially useful for parameter estimation. (Hurst index, drift, $\ldots$)

Thank you for your attention!