A stochastic mass conserved reaction-diffusion equation with nonlinear diffusion

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Mass conserved Allen-Cahn equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta \varphi + f(\varphi) - \dfrac{1}{|D|} \displaystyle\int_D f(\varphi) dx, & x \in D, \quad t \geq 0, \\ \nabla \varphi . n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ \varphi(0, x) = \varphi_0(x), & x \in D \end{cases}$$

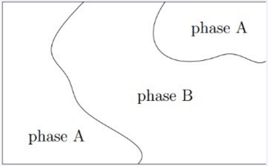
f has exactly 3 zeros $-1 < 0 < 1$ and

$$f'(\pm 1) < 0, \quad f'(0) > 0$$

A typical example is : $f(\varphi) = \varphi - \varphi^3$

Motivation

Deterministic mass conserved Allen-Cahn equation - linear diffusion.
Binary mixture undergoing phase separation.
(J. Rubinstein and P. Sternberg,IMA Journal of Applied Mathematics, 1992).



Stochastic Mass conserved Allen-Cahn equation

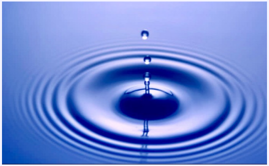
$$
\begin{cases}
\dfrac{\partial \varphi}{\partial t} = \Delta \varphi + f(\varphi) - \dfrac{1}{|D|} \displaystyle\int_D f(\varphi) dx + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\
\nabla \varphi . n = 0, & \text{on } \ \partial D \times \mathbb{R}^+ \\
\varphi(0, x) = \varphi_0(x), & x \in D
\end{cases}
$$

Motivation

Singular limit of the stochastic mass conserved equation - linear diffusion.

Motion of a droplet.

(D.C. Antonopoulou, P.W. Bates, D. Blömker and G.D. Karali, SIAM J. Math. Anal., 2016).



Our goal

Nonlocal Stochastic Reaction-Diffusion Equation with nonlinear diffusion

$$(P) \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = div(A(\nabla \varphi)) + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla \varphi).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ \varphi(0,x) = \varphi_0(x), & x \in D \end{cases}$$

Existence? uniqueness?

Our goal

- A is Lipschitz continuous from $\mathbb{R}^n$ to $\mathbb{R}^n$
- A is coercive

$$(A(a) - A(b))(a - b) \geq C_0(a - b)^2, \quad C_0 > 0$$

for all $a, b \in \mathbb{R}^n$.

(T. Funaki, H. Spohn,Communications in Mathematical Physics, 1997).

Remark:

$$\text{If } \; A = I \Rightarrow -div(A(\nabla u)) = -\Delta u.$$

Our goal

- The function $W(x,t)$ is a Q-Brownian motion in $L^2(D)$.

$$W(x,t)=\sum_{k=1}^{\infty}\beta_k(t)Q^{\frac{1}{2}}e_k(x)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k(x)$$

where:

1. $\{e_k\}_{k\geq 1}$ is an orthonormal basis in $L^2(D)$ diagonalizing Q.
2. $\{\lambda_k\}_{k\geq 1}$ are the corresponding eigenvalues for all $k\geq 1$.
3. Q is a nonnegative definite symmetric operator on $L^2(D)$ with $\mathrm{Tr}\,Q<+\infty$.

$$\mathrm{Tr}\,Q=\sum_{k=1}^{\infty}\langle Qe_k,e_k\rangle_{L^2(D)}=\sum_{k=1}^{\infty}\lambda_k\leq\Lambda_0.$$

4. $\{\beta_k(t)\}_{k\geq 1}$ is a sequence of independent $(\mathcal{F}_t)$-Brownian motions defined on $(\Omega,\mathcal{F},\mathrm{P})$.

A preliminary change of functions

We consider the nonlinear stochastic heat equation (N.V.Krylov and B.L.Rosovskii 2007)

$$(P_1) \quad \begin{cases} \dfrac{\partial W_A}{\partial t} = div(A(\nabla W_A)) + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla(W_A)).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ W_A(0,x) = 0, & x \in D \end{cases}$$

Definition

1. $W_A \in L^\infty(0,T;L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T);H^1(D))$; $div(A(\nabla W_A)) \in L^2(\Omega \times (0,T);(H^1(D))')$.
2. W_A *satisfies the integral equation*

$$W_A(x,t) = \int_0^t div(A(\nabla W_A(x,s)))ds + W(x,t)$$

- $W_A \in L^\infty(0,T;L^q(\Omega \times D))$ for all $q \in [2,\infty)$.

A preliminary change of functions

We define :

$$u(t) := \varphi(t) - W_A(t),$$

$$(P_2) \quad \begin{cases} \dfrac{\partial u}{\partial t} = div(A(\nabla(u + W_A)) - A(\nabla W_A)) + f(u + W_A) \\ \qquad\qquad - \dfrac{1}{|D|}\displaystyle\int_D f(u + W_A)dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u + W_A)).n = 0, & \text{on } \ \partial D \times \mathbb{R}^+ \\ u(0, x) = \varphi_0(x), & x \in D \end{cases}$$

Remark: The conservation of mass property holds, namely

$$\int_D u(x, t)dx = \int_D \varphi_0(x)dx, \ \text{ a.s. for a.e. } t \in \mathbb{R}^+.$$

A preliminary change of functions

We work with the following spaces:

$$H = \left\{ v \in L^2(D), \int_D v = 0 \right\}, \quad V = H^1(D) \cap H \text{ and } Z = V \cap L^{2p}$$

Definition

1. $u \in L^\infty(0, T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0, T); H^1(D)) \cap L^{2p}(\Omega \times (0, T) \times D)$; $div[A\nabla(u + W_A)] \in L^2(\Omega \times (0, T); (H^1)')$
2. *u satisfies the integral equation a.s.:*

$$u(x,t) = \varphi_0(x) + \int_0^t div[A(\nabla(u + W_A)) - A(\nabla W_A)]ds + \int_0^t f(u + W_A) - \int_0^t \frac{1}{|D|} \int_D f(u + W_A)dxds$$

Existence of a solution of Problem (P_2)

Theorem

There exists a unique solution of Problem (P_2).

Proof:

We apply the Galerkin method. Denote:

- $0 < \gamma_1 < \gamma_2 \leq ... \leq \gamma_k \leq ...$ eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions.
- $w_k, k = 0, ...$ smooth unit eigenfunctions in $L^2(D)$.

Existence of a solution of Problem (P_2)

Lemma

The functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$, in particular:

$$\int_D w_j w_0 = 0 \quad \textit{for all } j \neq 0 \;\; \textit{and} \;\; w_0 = \frac{1}{\sqrt{|D|}}.$$

Proof.

We check that $\int_D w_j(x)dx = 0$ for all $j \neq 0$. □

Existence of a solution of Problem (P_2)

We look for an approximate solution of the form

$$u_m(x,t) - M = \sum_{i=1}^{m} u_{im}(t) w_i, \quad M = \frac{1}{|D|}\int_D \varphi_0(x)dx$$

which satisfies the equation:

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j$$
$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla w_j + \int_D f(u_m + W_A)w_j$$
$$-\frac{1}{|D|}\int_D \Big(\int_D f(u_m + W_A)dx\Big) w_j dx,$$

for all w_j, $j = 1, ..., m$.

$u_m(x,0) = M + \sum_{i=1}^{m}(\varphi_0, w_i)w_i$ converges strongly in $L^2(D)$ to φ_0 as $m \to \infty$.

Existence of a solution of Problem (P_2)

Remark: The contribution of the nonlocal term vanishes !!

$$\int_D w_j(x)dx = 0, \quad \text{for all} \;\; j \neq 0$$

$$\Downarrow$$

$$-\frac{1}{|D|}\int_D \Big(\int_D f(u_m + W_A)dx\Big) w_j = 0$$

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j$$
$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla w_j + \int_D f(u_m + W_A)w_j$$

for all w_j, $j = 1, ..., m$.

A priori estimates

Lemma

There exists a positive constante $\mathcal{C}$

$$
\begin{aligned}
\mathbb{E}\int_D (u_m(t)-M)^2 dx &\leq \mathcal{C}, \quad \textit{for all} \;\; t\in[0,T] \\
\mathbb{E}\int_0^T\int_D |\nabla(u_m-M)|^2 dxdt &\leq \mathcal{C} \\
\mathbb{E}\int_0^T\int_D (u_m-M)^{2p} dxdt &\leq \mathcal{C} \\
\mathbb{E}\int_0^T\int_D (f(u_m+W_A))^{\frac{2p}{2p-1}} &\leq \mathcal{C} \\
\mathbb{E}\int_0^T \|divA(\nabla(u_m+W_A))\|^2_{(H^1)'} &\leq \mathcal{C}
\end{aligned}
$$

A priori estimates

Hence there exist a subsequence which we denote again by $\{u_m - M\}$ and functions
$u - M \in L^2(\Omega \times (0, T); V) \cap L^{2p}(\Omega \times (0, T) \times D) \cap L^\infty(0, T; L^2(\Omega \times D))$,
χ and Φ such that:

$$\begin{aligned}
u_m - M &\rightharpoonup u - M \quad \text{weakly in } L^2(\Omega \times (0, T); V) \\
&\qquad\qquad \text{and } L^{2p}(\Omega \times (0, T) \times D) \\
u_m - M &\rightharpoonup u - M \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega \times D)) \\
f(u_m + W_A) &\rightharpoonup \chi \quad \text{weakly in } L^{\frac{2p}{2p-1}}(\Omega \times (0, T) \times D) \\
div(A(\nabla(u_m + W_A))) &\rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega \times (0, T); (H^1)')
\end{aligned}$$

as $m \to \infty$.

Passing to the limit

We pass to the limit as $m \to \infty$

$$\langle u(t) - M, w \rangle = \langle \varphi_0 - M, w \rangle + \int_0^t \langle \Phi - div(A(\nabla W_A)), w \rangle + \int_0^t \langle \chi, w \rangle$$

for all $w \in V \cap L^{2p}(D)$.

It remains to prove that :

$$\langle \Phi + \chi, w \rangle = \langle div(A(\nabla(u + W_A))) + f(u + W_A(t)), w \rangle \quad \text{for all } w \in V \cap L^{2p}(D).$$

Monotonicity argument

(M.Marion 1987- N.V.Krylov and B.L.Rosovskii 2007)

- For the nonlinear diffusion term use coercivity !!
- For the reaction term use change of function !!
- Nonlocal term vanishes !!

Uniqueness

Proof:

- Let u_1 and u_2 be two solutions of Problem (P_2)

$$\begin{aligned} u_1(t) - u_2(t) &= \int_0^t div(A(\nabla(u_1 + W_A) - A(\nabla(u_2 + W_A)) \\ &+ \int_0^t [f(u_1 + W_A) - f(u_2 + W_A)] \\ &- \frac{1}{|D|} \int_0^t [\int_D f(u_1 + W_A) - \int_D f(u_2 + W_A) dx]. \end{aligned}$$

Uniqueness

- Taking the duality product with $u_1 - u_2$
- Same initial condition $u_1(x,0) = u_2(x,0) = \varphi_0(x) \Rightarrow$
$$-\frac{1}{|D|}\int_0^t [\int_D f(u_1 + W_A) - \int_D f(u_2 + W_A)] \int_D (u_1 - u_2) = 0.$$
- Taking the expectation of the equation
$$\mathbb{E}\int_D (u_1 - u_2)^2(x,t)dx \leq C_6 \mathbb{E}\int_0^t \int_D (u_1 - u_2)^2(x,t)dxds,$$

By Gronwall's Lemma

$$u_1 = u_2 \quad \text{a.e. in } \Omega \times D \times (0,T).$$

THANK YOU FOR YOUR ATTENTION !

Brownian motion

Brownian motion is described by the Wiener process. The Wiener process W_t is characterised by four facts:

1. $W_0 = 0$.
2. W_t is almost surely continuous.
3. $W_t$has independent increments means that if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables.
4. $W_t - W_s \sim \mathcal{N}(0, t-s)$(for $0 \leq s \leq t$).

$\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

Appendix

Theorem

Let W_A be a solution of Problem (P_1) there holds $W_A \in L^\infty(0, T, L^{2q}(\Omega \times D))$, for all $q \geq 1$.

For any positive k, denote by Φ_k the even function such that

$$\Phi_k(x) = \begin{cases} x^{2q}, \\ q(2q-1)k^{2(q-1)}x^2 - 4q(q-1)k^{2q-1}x + (q-1)(2q-1)k^{2 \end{cases}$$

Φ_k is a C^2-convex function and Φ'_k is a Lipschitz-continuous function with $\Phi'_k(0)$=0. Thus, for any positive x, one gets $0 \leq \Phi'_k(x) \leq c(k)x$ and $0 \leq \Phi_k(x) = \int_0^x \Phi'_k(s)ds \leq \frac{c(k)}{2}x^2$. This yields that, $\mathbb{E}\int_D \Phi_k(W_A(x,t))dx \leq \frac{c(k)}{2}\mathbb{E}\int_D W_A^2(x,t)dx \leq \overline{c}(k)$ for a.e. $t \in [0, T]$.

Lemma

One has $0 \leq \Phi''_k(x) \leq 2q(2q-1)(1 + \Phi_k(x))$, for all $x \in \mathbb{R}$.

Appendix

Taking inspiration from Debussche, Hofmanova and Vovelle we prove a generalized Itô Formula for weak solutions of Problem (P_1).

Proposition

Let W_A be the solution of Problem (P_1). Then almost surely, for all $t \in [0, T]$.

$$\begin{aligned} \int_D \Phi_k(W_A(t))dx &= \int_0^t \int_D \Phi'_k(W_A(s))div(A(\nabla W_A(s)))dxds \\ &+ \int_0^t \int_D \Phi'_k(W_A(s))dW(s) \\ &+ \frac{1}{2}\sum_{l=1}^{\infty} \int_0^t \int_D \Phi''_k(W_A)\lambda_l e_l^2 dxds \end{aligned} \tag{1}$$

Itô's Formula

Definition

Let Φ be an H-valued process stochastically integrable process, let h be an H-valued progressively measurable Bochner integrable process and $X(0)$ an $\mathcal{F}_0$ measurable H-valued random variable. Consider the following well defined process :

$$X(t) = X(0) + \int_0^t h(s)ds + \int_0^t \Phi(s)dW(s), t \in [0, T].$$

Assume a function $F : H \to \mathbb{R}$ and its partial derivatives F_x, F_{xx}, are uniformly continuous on bounded subsets of H. Under the above conditions, $\mathbb{P}$-a.s., for all $t \in [0, T]$.

$$\begin{aligned} F(X(t)) &= F(X(0)) + \int_0^t \langle F_x(X(s)), h(s)\rangle ds \\ &+ \int_0^t \langle F_x(X(s)), \Phi(s)dW(s)\rangle_H + \frac{1}{2}\int_0^t Tr(F_{xx}(X(s))(\Phi(s)\sqrt{Q} \end{aligned}$$

Appendix

Taking the expectation of (2) one has that

$$\mathbb{E}\int_D \Phi_k(W_A)dx \leq -C_0\mathbb{E}\int_0^t\int_D \Phi_k''(W_A)|\nabla W_A|^2 + \frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k''(W_A)c$$

$$\leq \frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k''(W_A)dxds$$

Then using the previous Lemma and Gronwall Lemma yield

$$\mathbb{E}\int_D \Phi_k(W_A))dx \leq C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}$$

Thus, $\mathbb{E}\int_D \Phi_k(W_A)dx$ is bounded independently of k.
Finally, $\Phi_k(x)$ by monotone convergence theorem we conclude that

$$\lim_{k\to\infty}\mathbb{E}\int_D \Phi_k(W_A(t)))dx \leq C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}$$

for all $t > 0$.

Adapted process

In the study of stochastic processes, an adapted process (also referred to as a non-anticipating or non-anticipative process) is one that cannot "see into the future". An informal interpretation is that X is adapted if and only if, for every realisation and every n, X_n is known at time n.
Definition
Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space;
- I be an index set with a total order $\leq$ (often, I is $\mathbb{N}, \mathbb{N}_0$, $[0, T]$ or $[0, +\infty)$) ;
- $\mathcal{F}_\cdot = (\mathcal{F}_i)_{i \in I}$be a filtration of the sigma algebra $\mathcal{F}$;
- (S, Σ) be a measurable space, the state space;
- $X : I \times \Omega \to S$ be a stochastic process.

The process X is said to be adapted to the filtration $(\mathcal{F}_i)_{i \in I}$ if the random variable $X_i : \Omega \to S$ is a $(\mathcal{F}_i, \Sigma)$-measurable function for each $i \in I$.

Our goal

- D is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂D;
- The nonlinear function f is a smooth function which satisfies the following properties:

(F_1) There exist positive constants C_1 and C_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2(b^{2p}+1), \text{ for all } a, b \in \mathbb{R}$$

(F_2) There exists a positive constant $C_4 = C_4(M)$ such that

$$|f(s)| \leq C_4(|s-M|^{2p-1}+1)$$

(F_3) There exists a positive constant C_6 such that

$$f'(s) \leq C_6.$$

In particular, one could take $f(s) = \sum_{j=0}^{2p-1} b_j s^j$ with $b_{2p-1} < 0, p \geq 2$.

Our goal

- A is Lipschitz continuous from $\mathbb{R}^n$ to $\mathbb{R}^n$, $A(0) = 0$ and

$$|A(a) - A(b)| \leq C|a - b|, \quad C > 0 \tag{2}$$

- A is coercive

$$(A(a) - A(b))(a - b) \geq C_0(a - b)^2, \quad C_0 > 0 \tag{3}$$

for all $a, b \in \mathbb{R}^n$.

(T. Funaki, H. Spohn,Communications in Mathematical Physics, 1997).

Remark:

$$\text{If } \; A = I \Rightarrow -div(A(\nabla u)) = -\Delta u.$$

A priori estimates

- Multiply the equation of u_m by $u_{jm} = u_{jm}(t)$ and sum on $j = 1, ..., m$

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)(u_m - M)$$
$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla(u_m - M)$$
$$+ \int_D f(u_m + W_A)(u_m - M)$$

- Coercivity property of A to bound the generalized Laplacian term

$$-\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla(u_m - M)$$
$$\leq -C_0 \int_D |\nabla(u_m - M)|^2 dx$$

A priori estimates

Using the property F_1 we deduce that

$$\begin{aligned}\int_D f(u_m + W_A(t))(u_m - M) &= \int_D f(u_m - M + M + W_A(t))(u_m - M)\\ &\leq -\int_D C_1(u_m - M)^{2p} + C_2\int_D |W_A(t)|^{2p}\\ &\quad +C_2|D|,\end{aligned}$$

Integrating from 0 to T and taking the expectation :

$$\begin{aligned}&\mathbb{E}\int_D (u_m(T) - M)^2 dx + 2C_0\mathbb{E}\int_D |\nabla(u_m - M)|^2 dx\\ &+2C_1\mathbb{E}\int_D (u_m - M)^{2p}\\ &\leq \int_D (u_m(0) - M)^2 dx + 2C_2\mathbb{E}\int_0^T\int_D |W_A(t)|^{2p} + 2C_2|D|T\\ &\leq K\end{aligned}$$

Monotonicity argument

(M.Marion 1987- N.V.Krylov and B.L.Rosovskii 2007)
Let $w(t,\omega)$ be any measurable function in (t,ω) with values in $H^1(D) \cap L^{2p}(D)$.

$$\begin{aligned}
\mathcal{O}_m &= \mathbb{E}\Big[\int_0^T e^{-cs}\{2\langle div\big(A(\nabla(u_m - M + W_A))\big) \\
&\qquad - div\big(A(\nabla(w - M + W_A))\big), u_m - M - (w - M)\rangle \\
&\quad + 2\langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M)\rangle \\
&\quad - c\|u_m - M - (w - M)\|^2\} ds \\
&= J_1 + J_2 + J_3
\end{aligned}$$

Using the coercivity property of A and $f' \leq C_6$ we prove that :

$$\mathcal{O}_m \leq 0, \quad \text{choosing } c \text{ large enough.}$$

Monotonicity argument

For J_1 use the coercivity property

$$\begin{aligned} J_1 &= -2\mathbb{E}\int_0^T e^{-cs}\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(w - M + W_A))] \\ &\qquad [u_m - M - (w - M)] \\ &\leq -2C_0\mathbb{E}\int_0^T e^{-cs}\|u_m - w\|^2 ds \\ &\leq 0 \end{aligned}$$

Using (F_3) and mean value theorem :

$$\begin{aligned} J_2 &= \mathbb{E}\int_0^T e^{-cs}\langle f(u_m + W_A) - f(w + W_A), u_m - w\rangle ds \\ &\leq \mathbb{E}\int_0^T e^{-cs} C_6\|u_m - w\|^2 ds \end{aligned}$$

Choosing $c \geq C_6$ we get our result.

Monotonicity

We choose $w - M = u - M - \lambda v$, $\lambda \in \mathbb{R}_+$ such that that $v(t,\omega)$ is any measurable process with values in $V \cap L^{2p}(D)$.

$$\mathbb{E}\int_0^T e^{-cs}\langle \Phi + \chi - div(A\nabla(u - M + W_A)) - f(u + W_A), v\rangle dt \leq 0$$

Since v is arbitrary, it follows that

$$\langle \Phi + \chi, v\rangle \quad = \quad \langle div(A\nabla(u - M + W_A)) + f(u + W_A), v\rangle,$$

for all $v \in V \cap L^{2p}(D)$.

We conclude that a.s.:

$$u(x,t) \quad = \quad \varphi_0(x) + \int_0^t div[A(\nabla(u + W_A)) - A(\nabla W_A)]ds + \int_0^t f(u + W_A)$$
$$- \int_0^t \frac{1}{|D|}\int_D f(u + W_A)dxds$$