Spatial recurrences, associated models, limit shapes

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UPMC – LPMA

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Les probabilités de demain
IHÉS

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Desnanot-Jacobi identity

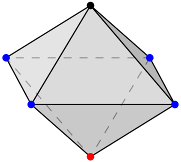
$$\det(M)\det(M_{1,n}^{1,n}) = \det(M_1^1)\det(M_n^n) - \det(M_1^n)\det(M_n^1)$$

M is a square matrix of size n.
M_I^J is the matrix obtained by removing rows in I and columns in J.

$$\begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \vdots & \vdots & & \vdots \\ m_{n,1} & m_{n,2} & \dots & m_{n,n} \end{pmatrix}$$

The octahedron recurrence

$$x_{i,j}^{(n+1)} = \frac{x_{i-1,j}^{(n)} x_{i+1,j}^{(n)} - x_{i,j-1}^{(n)} x_{i,j+1}^{(n)}}{x_{i,j}^{(n-1)}}$$



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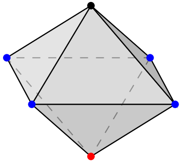

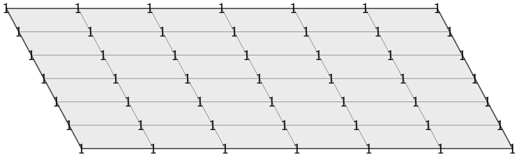

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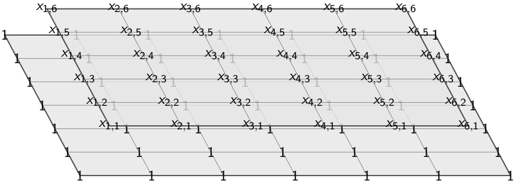

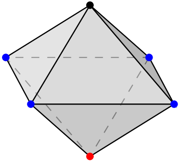

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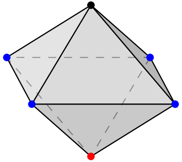

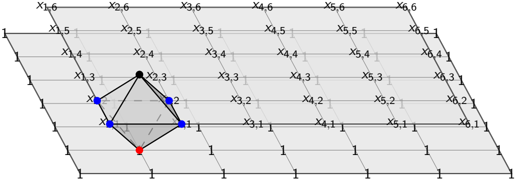

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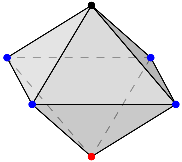

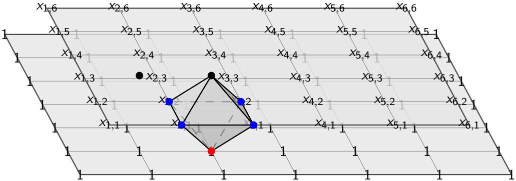

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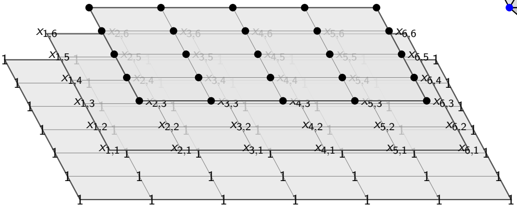

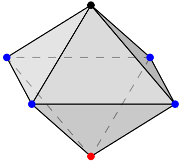

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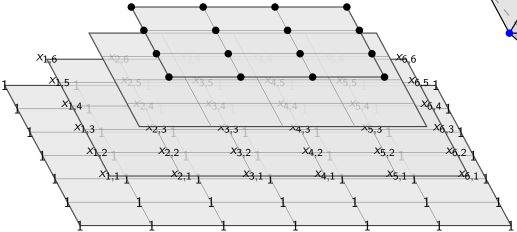

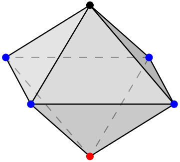

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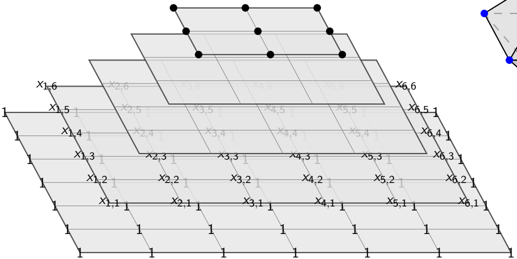

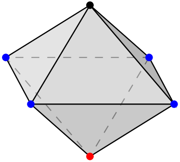

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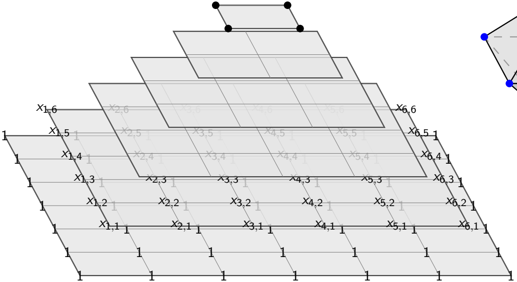

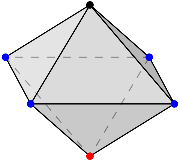

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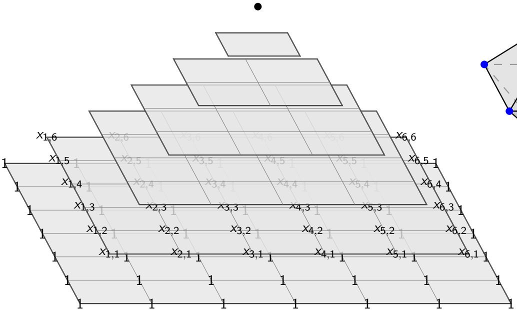

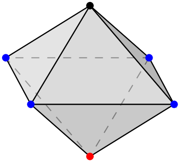

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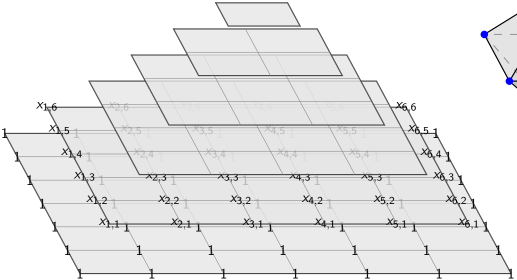

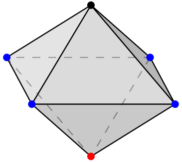

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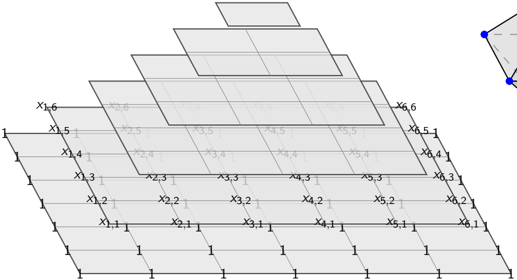

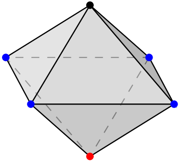

Dodgson, C. L. "Condensation of Determinants, Being a New and Brief Method for Computing Their Arithmetical Values." *Proceedings of the Royal Society of London* 15 (1866): 150-55.

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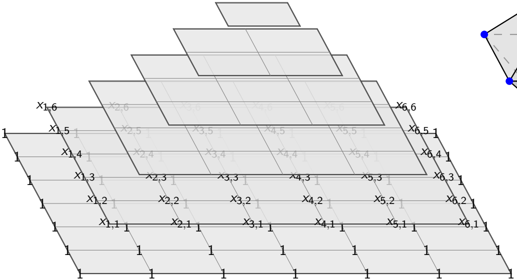

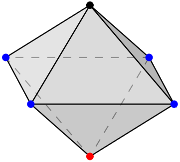

Question

… what about other initial conditions?

The octahedron recurrence

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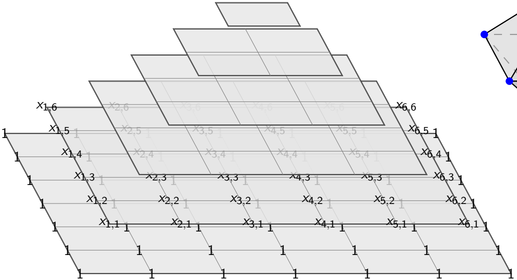
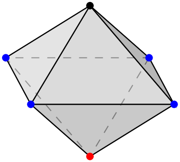




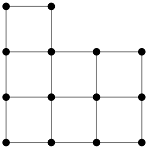
Question

... what about other initial conditions?

Result of experiments: the solution seems to be a *Laurent polynomial* in the initial conditions.

The dimer model

$G = (V, E)$
a planar graph, with positive $(\mu_e)_{e \in E}$.



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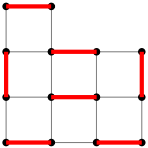

Dimers configuration: subset of edges $m \subset E$ such that every vertex is adjacent to exactly one edge $e \in m$.

$$\mathbb{P}(m) = \frac{1}{\mathcal{Z}_{dim}} \prod_{e \in m} \mu_e.$$

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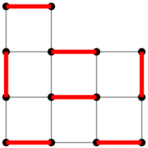

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$$\mathcal{Z}_{dim} = \sum_m \prod_{e \in m} \mu_e$$

is the **partition function**.

Solving the octahedron recurrence

Recap

$G = (V, E)$. $(\mu_e)_{e \in E}$.

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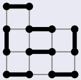
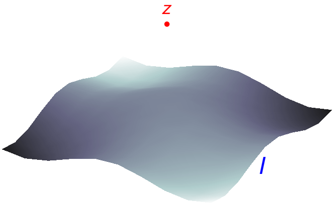


We fix initial conditions $(f_i)_{i \in I}$ on $I \subset \mathbb{Z}^3$, and some z. Suppose that using the octahedron recurrence we can define f_z.



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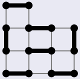

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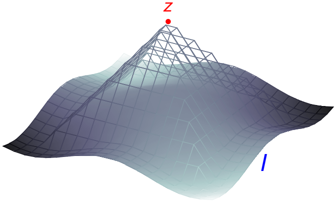

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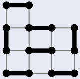
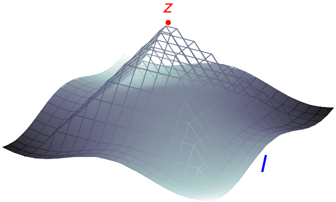


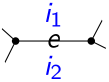
We fix initial conditions $(f_i)_{i \in I}$ on $I \subset \mathbb{Z}^3$, and some z.
Suppose that using the octahedron recurrence we can define f_z.
[Speyer 2004]
There is a graph $G(I, z)$ such that

$$f_z \simeq \mathcal{Z}_{dim}(G)$$



The faces of G are in bijection with I and the weights $(\mu_e)_{e \in E}$ are given by

$$\mu_e = \frac{1}{f_{i_1} f_{i_2}}, \quad \text{where}$$



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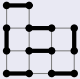

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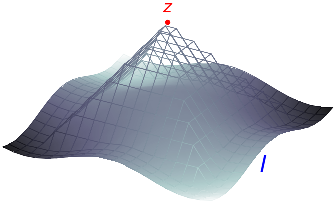

$$f_z = \mathcal{Z}_{dim}(G) \times \prod_{i \in I \cap C_z} f_i^{\lceil \frac{\deg(i)}{2} \rceil - 1}$$

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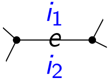
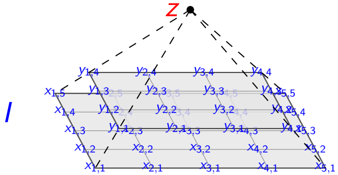


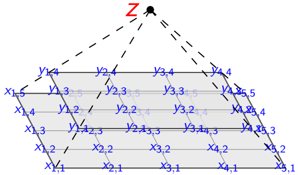
Example



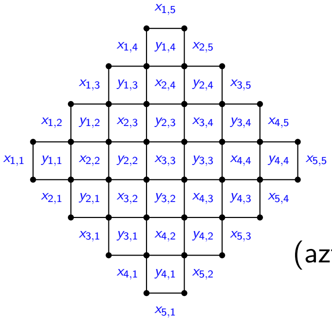
Z
I
y1,4 y2,4 y3,4 y4,4
y1,3 y2,3 y3,3
y1,2 y2,2 y3,2
x1,5 x1,4 x1,3 x1,2 x1,1
x2,2 x3,2 x4,2 x5,2
x2,1 x3,1 x4,1 x5,1

Example

I

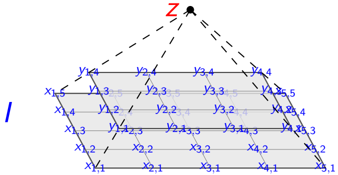


$G =$



(aztec diamond)

Example



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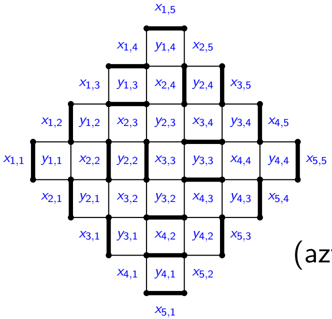

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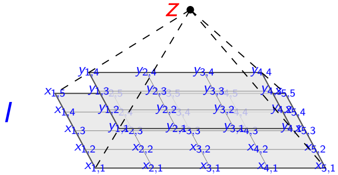

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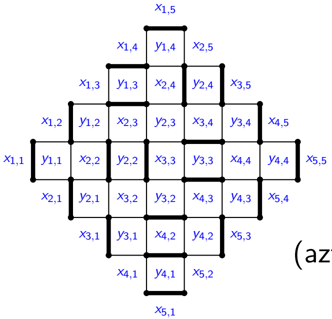

$f_z \simeq \mathcal{Z}_{dim}(G)$

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Solving the octahedron recurrence

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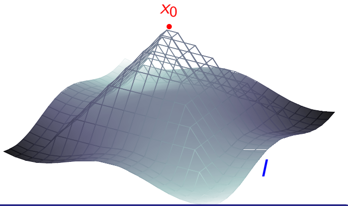


Algebraic consequences

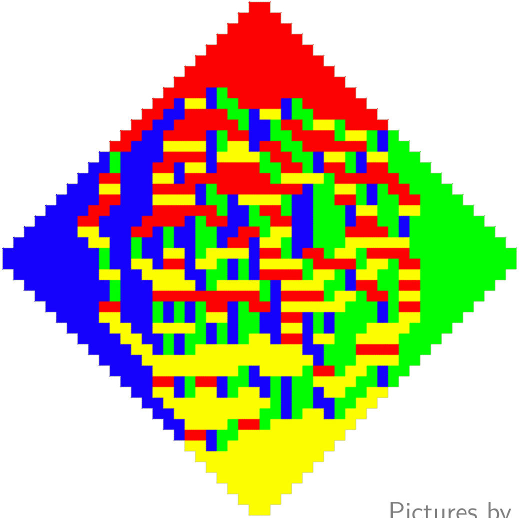
- The solution of the octahedron recurrence is a **Laurent polynomial** in the initial conditions:

$$f_z \in \mathbb{Z}[(f_i^{\pm 1})_{i \in I}].$$

[Fomin - Zelevinsky, 2001]

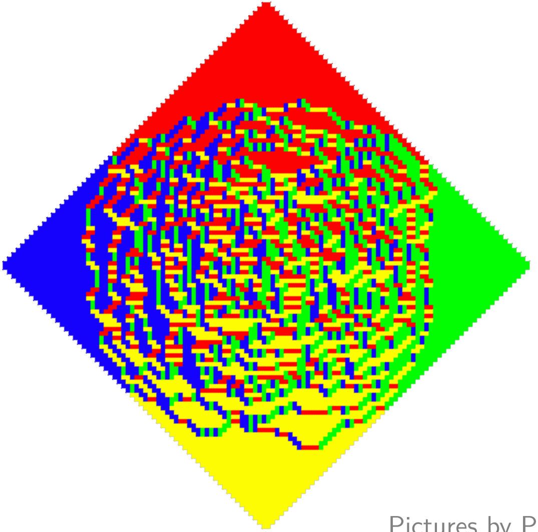
- The coefficients are 1.
- The exponents of the f_i are in $\{-1, \dots, 3\}$.

Limit shape



Pictures by Patrik Ferrari

Limit shape



Pictures by Patrik Ferrari

Where do these limit shape come from? (*Di Francesco, Soto-Garrido*)

Using only the fact that the partition function $\mathcal{Z}$ satisfies the octahedron recurrence:

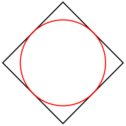
$$\mathcal{Z}_{i,j,k+1}\mathcal{Z}_{i,j,k-1} = \mathcal{Z}_{i-1,j,k}\mathcal{Z}_{i+1,j,k} + \mathcal{Z}_{i,j-1,k}\mathcal{Z}_{i,j+1,k},$$

and techniques of Baryshnikov, Pemantle, Wilson, it is possible to find the behaviour of

$$\rho_z = \mathbb{E}\left[1 - \#\{\text{dim. around } z\}\right].$$

For $z = (ku, kv, k)$,

- If $2(u^2 + v^2) > 1$,
  $\rho_{ku,kv,k} \to 0$ exponentially in k.
- If $2(u^2 + v^2) < 1$, $\rho_{ku,kv,k} \sim \frac{2}{\pi k \sqrt{1-2(u^2+v^2)}}$.

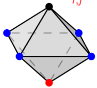


arctic circle

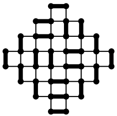
To sum things up...

$$x_{i,j}^{(n+1)} = \frac{x_{i-1,j}^{(n)} x_{i+1,j}^{(n)} + x_{i,j-1}^{(n)} x_{i,j+1}^{(n)}}{x_{i,j}^{(n-1)}}$$

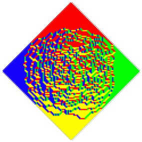
A nice **spatial recurrence**,



A **model** such that $\mathcal{Z}$ solves the recurrence,



Free **limit shapes**.



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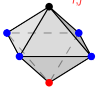

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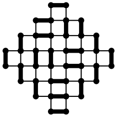

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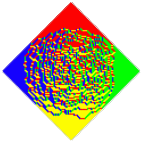

Question

What are some nice recurrences?

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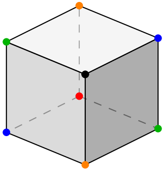
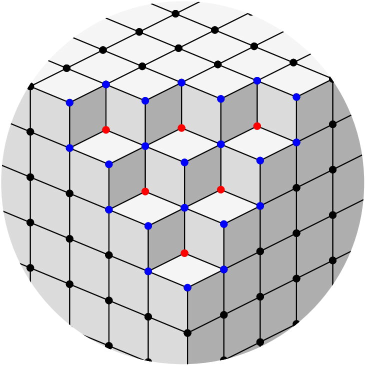
The cube recurrence

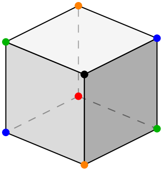


$$f_{i,j,k} f_{i+1,j+1,k+1} = f_{i+1,j,k} f_{i,j+1,k+1} + f_{i,j+1,k} f_{i+1,j,k+1} + f_{i,j,k+1} f_{i+1,j+1,k}$$

- Suggested by Propp in 2001,
- Related to the *star-triangle* transformation of resistor networks.

The cube recurrence





$$f_{i,j,k} f_{i+1,j+1,k+1} = f_{i+1,j,k} f_{i,j+1,k+1} + f_{i,j+1,k} f_{i+1,j,k+1} + f_{i,j,k+1} f_{i+1,j+1,k}$$

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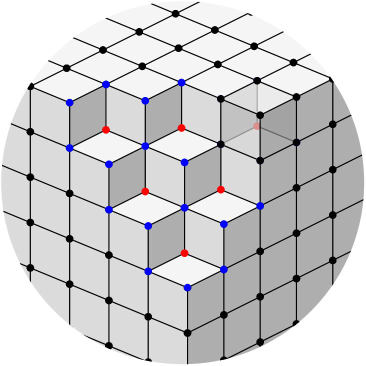

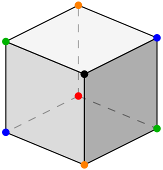

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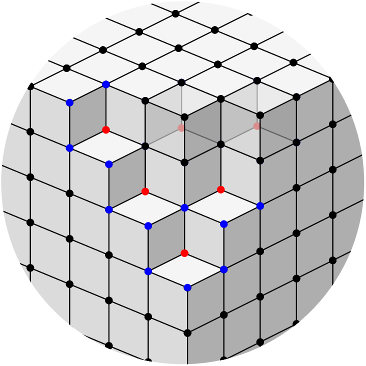

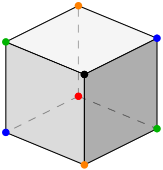

$$f_{i,j,k} f_{i+1,j+1,k+1} = f_{i+1,j,k} f_{i,j+1,k+1} + f_{i,j+1,k} f_{i+1,j,k+1} + f_{i,j,k+1} f_{i+1,j+1,k}$$

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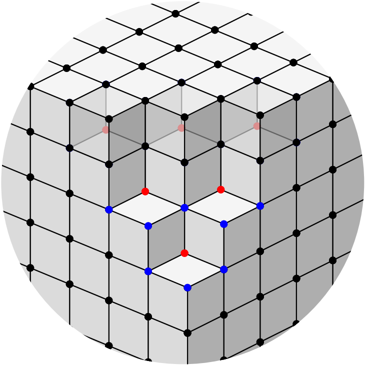

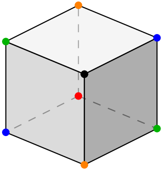

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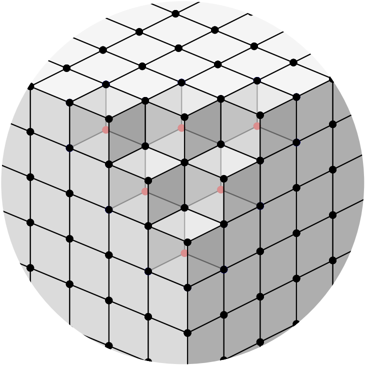

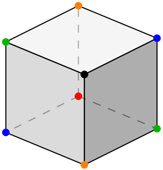

$$f_{i,j,k}f_{i+1,j+1,k+1} = f_{i+1,j,k}f_{i,j+1,k+1}$$
$$+f_{i,j+1,k}f_{i+1,j,k+1}$$
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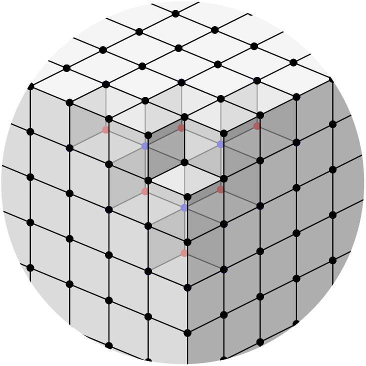

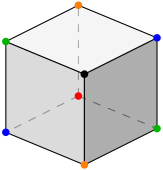

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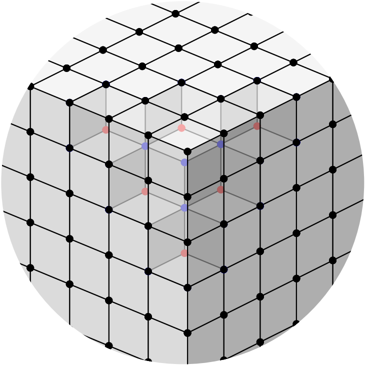

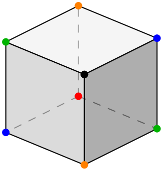

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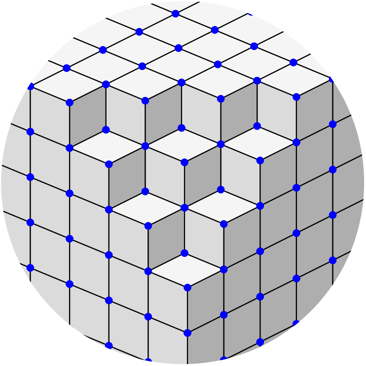
Laurentness property!

Question

Is there a hidden model behind this recurrence?

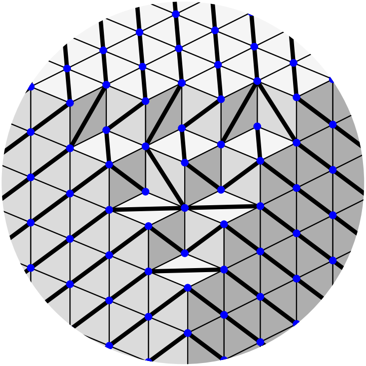
Cube groves *[Carroll - Speyer, 2004]*

On a stepped
surface (pile of cubes),



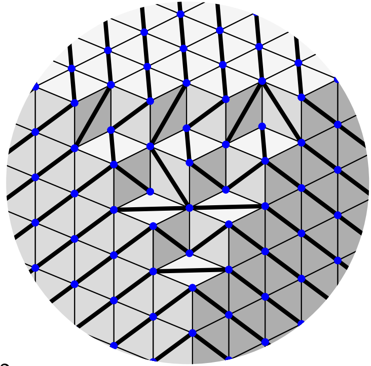
Cube groves *[Carroll - Speyer, 2004]*

On a stepped surface (pile of cubes), a *grove* **g** is a choice of diagonals of each face that give a **spanning forest**, with some boundary/connectivity conditions.



Cube groves *[Carroll - Speyer, 2004]*

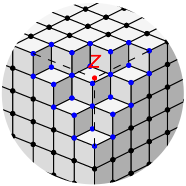
On a stepped surface (pile of cubes), a *grove* **g** is a choice of diagonals of each face that give a **spanning forest**, with some boundary/connectivity conditions.
We equip the vertices with weights (f_i).



$$\mathbb{P}(\mathbf{g}) = \frac{1}{\mathcal{Z}} \prod_i f_i^{\deg_{\mathbf{g}}(i)-2}$$

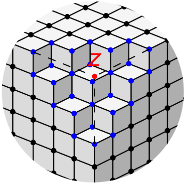
$$\mathcal{Z} = \sum_{\mathbf{g}} \prod_i f_i^{\deg_{\mathbf{g}}(i)-2}.$$

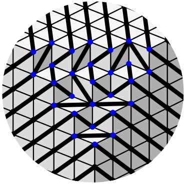
Solving the cube recurrence



We fix initial conditions $(f_i)_{i \in I}$ on $I \subset \mathbb{Z}^3$. Suppose there is a $z \in \mathbb{Z}^3$ so that the cube recurrence lets us define f_z,

Solving the cube recurrence





We fix initial conditions $(f_i)_{i\in I}$ on $I \subset \mathbb{Z}^3$. Suppose there is a $z \in \mathbb{Z}^3$ so that the cube recurrence lets us define f_z,then

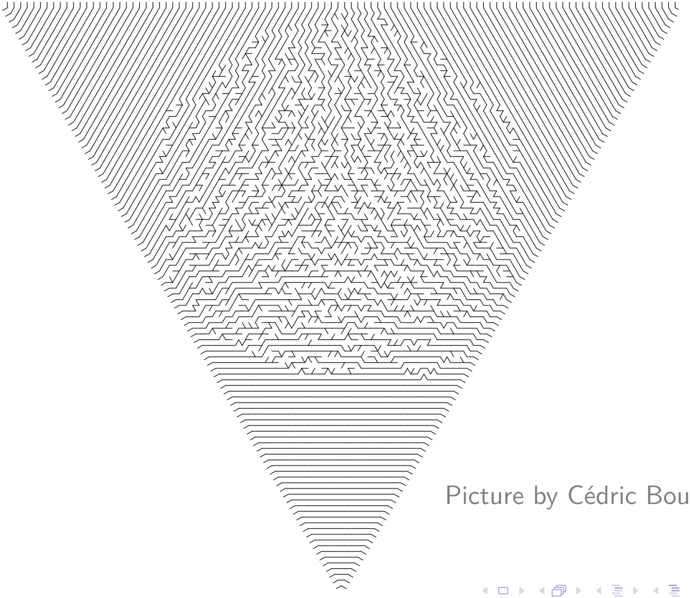
Theorem (*Carroll, Speyer*)

$$f_z = \mathcal{Z}$$

where $\mathcal{Z}$ is the partition function of groves on I with weights (f_i):

$$\mathcal{Z} = \sum_{\mathbf{g}} \prod_i f_i^{\deg_{\mathbf{g}}(i)-2}.$$

Limit shapes *[Petersen - Speyer, 2004]*



Picture by Cédric Boutillier

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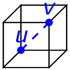
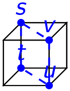
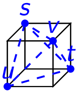
Kashaev's recurrence

$$\sum_{\text{diagonals}} f_u^2 f_v^2 \; - \; 2 \sum_{\text{rectangles}} f_s f_t f_u f_v \; - \; 4 \sum_{\text{tetrahedra}} f_s f_t f_u f_v = 0.$$







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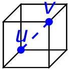
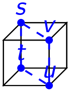
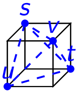


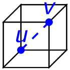


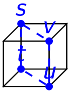


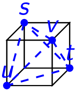
- Introduced by Kashaev in 1996 to study the *star-triangle* relation for the Ising model,
- Relation between minors $\left(\det(M_I^I)\right)_{I \subset \{1,2,3\}}$ for a *symmetric* matrix M (*Kenyon, Pemantle*),
- Defines f_v as a function of the others up to the choice of a root; we take the greater root,
- "Laurentness"!

Kashaev's recurrence

$$\sum_{\text{diagonals}} f_u^2 f_v^2 \; - \; 2 \sum_{\text{rectangles}} f_s f_t f_u f_v \; - \; 4 \sum_{\text{tetrahedra}} f_s f_t f_u f_v = 0.$$







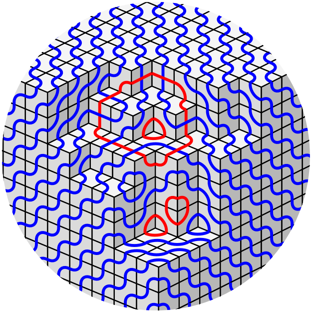
Question

Is there a hidden model behind this recurrence?

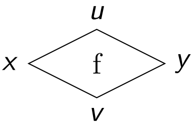
$C_2^{(1)}$ loop model

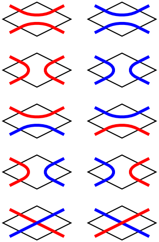
Bicolor loops model
introduced in *[Warnaar, Nienhuis 1993]*.

On a stepped
solid, put weights (g_x) on vertices.



Possible configurations at a face:





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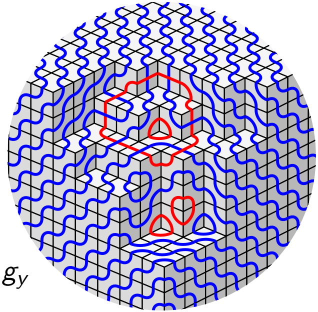

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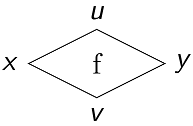

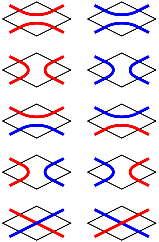

$g_x g_y$

$g_u g_v$

$\sqrt{g_x g_y}\sqrt{g_x g_y + g_u g_v}$

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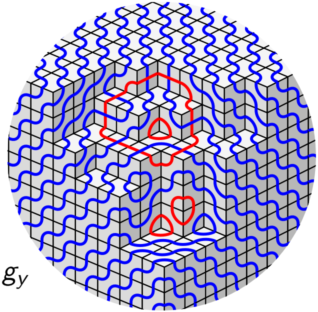

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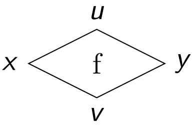

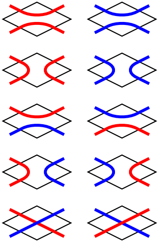

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$\sqrt{g_x g_y g_u g_v}$

Probability of a loop configuration proportional to $2^N \prod_{\mathrm{f}} \mathrm{weight}(\mathrm{f})$
(where N is the number of finite loops).

Solving Kashaev's recurrence

$$\mathcal{Z} = \sum_{C_2^{(1)} conf.} \left(2^N \prod_{\mathrm{f}} \mathrm{weight}(\mathrm{f}) \prod_{v} g_v^{-2} \right)$$

is the solution to Kashaev's recurrence in terms of the initial conditions $(g_v)_{v \in I}$.

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Consequences

- Laurentness in the g_v and $\sqrt{g_x g_y + g_u g_v}$ variables; bounds on exponents. The coefficients are powers of 2.

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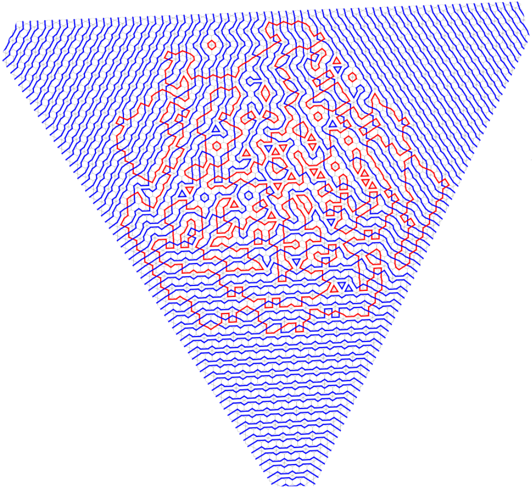
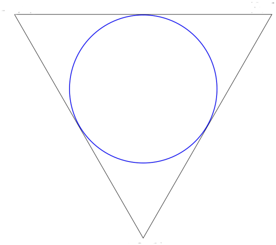
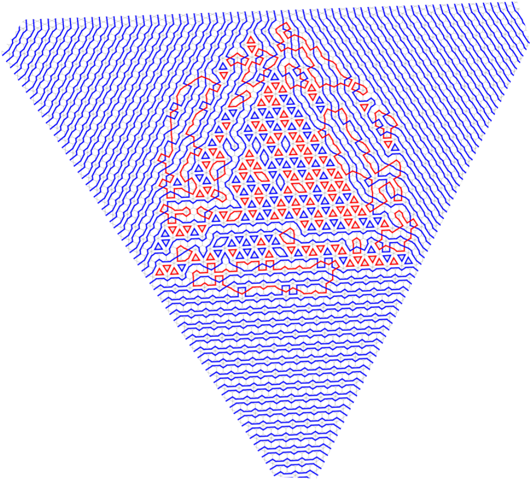
- Laurentness in the g_v and $\sqrt{g_x g_y + g_u g_v}$ variables; bounds on exponents. The coefficients are powers of 2.
- ...limit shapes!

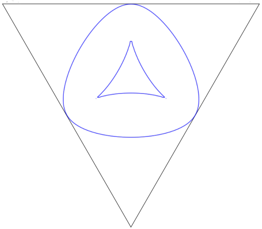
For periodic initial conditions,





For periodic initial conditions,





Known recurrences

Recurrence	**Model**
octahedron	dimers (*Speyer*)
cube	groves (*Carroll, Speyer*)
hexahedron	double-dimers (*Kenyon, Pemantle*)
Kashaev	$C_2^{(1)}$ loops (*M.*)

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Disclaimer

There is probably more to that list!

Merci

Question

Where do these limit shapes come from?

Jockush, Propp, Shor 1998 ; Di Francesco, Soto-Garrido 2014

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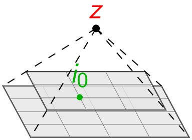

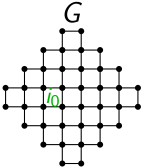

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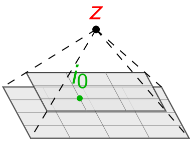

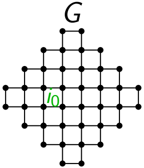

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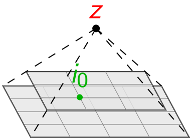

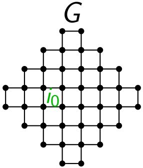

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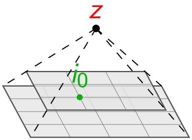

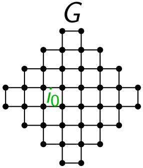

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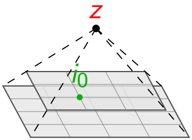

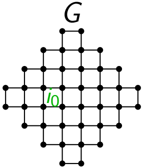

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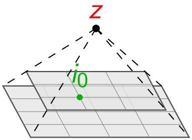

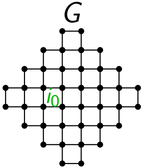

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(i_0 is fixed)

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Observable ρ_z

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$$\rho_{i,j,k+1} + \rho_{i,j,k-1} = \frac{1}{2}\left(\rho_{i-1,j,k} + \rho_{i+1,j,k} + \rho_{i,j-1,k} + \rho_{i,j+1,k}\right).$$

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- Generating function $g(x,y,z) = \sum_{i,j,k} \rho_{i,j,k} x^i y^j z^k$:

$$g(x,y,z) = \frac{z}{1 + z^2 - \frac{z}{2}(x^{-1} + x + y^{-1} + y)}.$$

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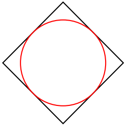
Analysis of the singularity of g at $(1,1,1)$ (*Baryshnikov, Pemantle, Wilson*): for $(i,j,k) = (ku, kv, k)$,

- If $2(u^2 + v^2) > 1$,
  $\rho_{ku,kv,k} \to 0$ exponentially in k.
- If $2(u^2 + v^2) < 1$, $\rho_{ku,kv,k} \sim \frac{2}{\pi k \sqrt{1 - 2(u^2 + v^2)}}$.



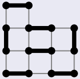
arctic circle

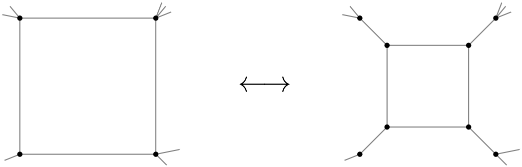
Spider move

Recap

$G = (V, E)$. $(\mu_e)_{e \in E}$.

$$\mathbb{P}(m) = \frac{1}{\mathcal{Z}_{dim}} \prod_{e \in m} \mu_e, \quad \mathcal{Z}_{dim} = \sum_m \prod_{e \in m} \mu_e.$$





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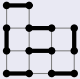

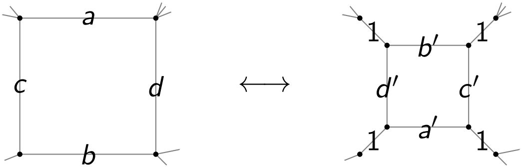

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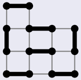
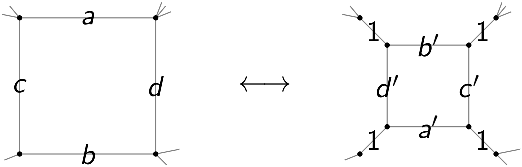
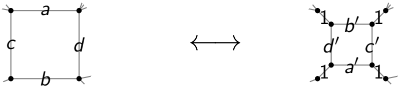




The dimer measure is preserved *iff* :

$$a' = \frac{a}{ab+cd} \qquad c' = \frac{c}{ab+cd}$$

$$b' = \frac{b}{ab+cd} \qquad d' = \frac{d}{ab+cd}$$



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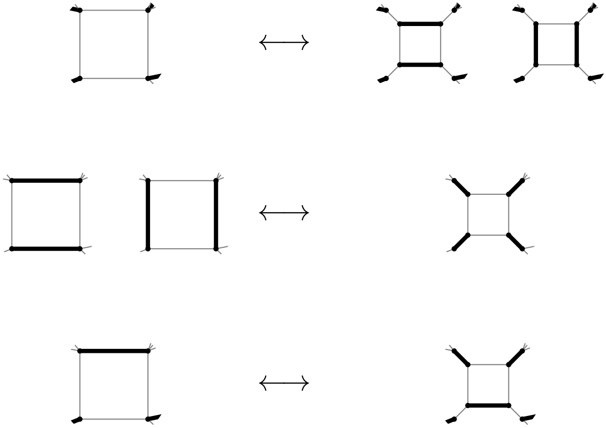
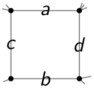
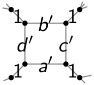


. . .



$\longleftrightarrow$



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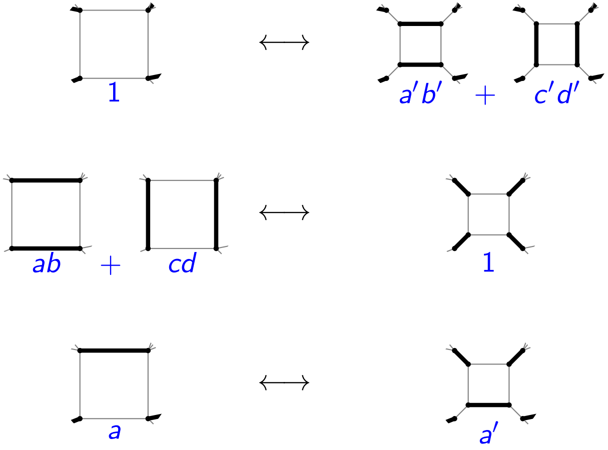

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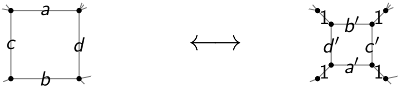

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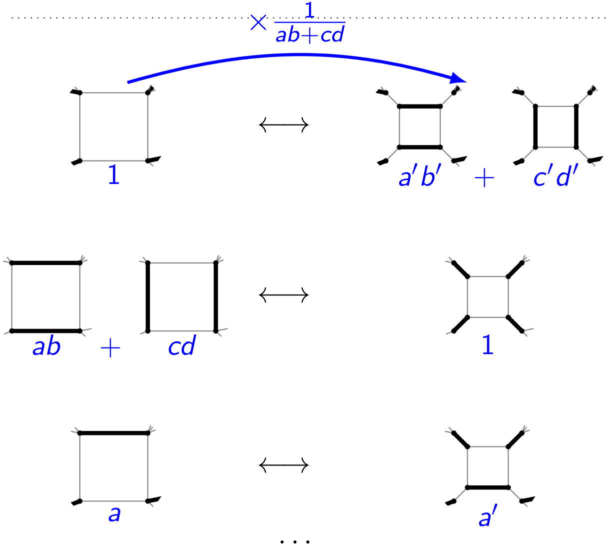

Spider move

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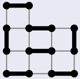
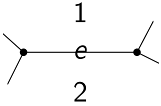


If μ takes the form

$$\mu_e = \frac{1}{A_1 A_2}$$



with $(A_f)_{f \in F}$ variables on the faces of G, then

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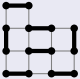

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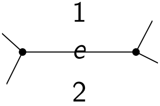

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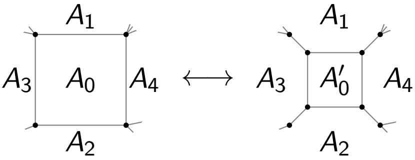

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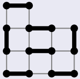

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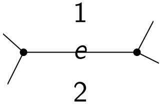

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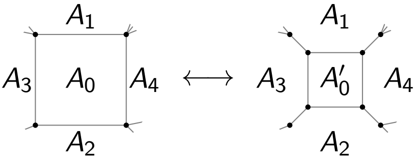
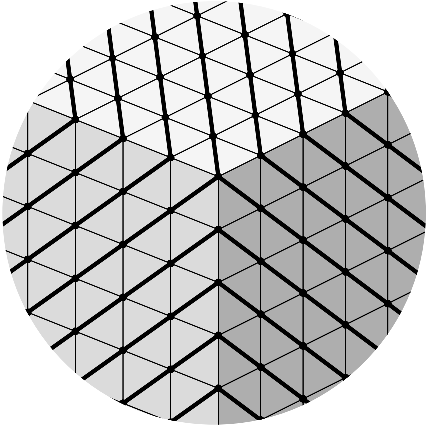
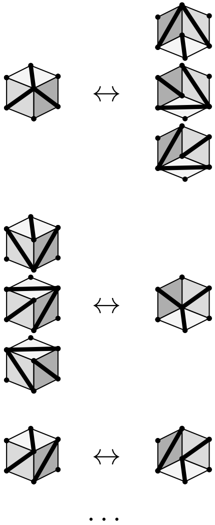


iff $\quad A'_0 = \frac{A_1 A_3 + A_2 A_4}{A_0}$

Initial condition:



Coupling:



Coupling:

It is indeed
a coupling *iff* f satisfies
the cube recurrence.