GENE FLOW ACCROSS A GEOGRAPHICAL BARRIER

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Les Probabilités de demain
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GEOGRAPHICAL BARRIERS TO DISPERSAL

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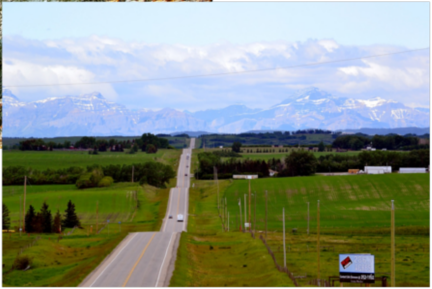

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STEPPING STONE MODEL OF DISPERSAL

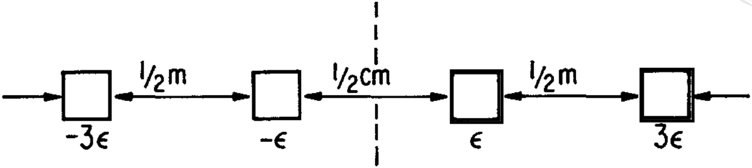


Figure: Stepping stone model with a barrier

from Nagylaki 1976

At each generation,

- the N individuals in each colony are replaced by new individuals
- a proportion $1 - m$ of them are the offspring of (uniformly chosen) parents in the same colony,
- a proportion m are the offspring of parents in neighbouring colonies

STEPPING STONE MODEL OF DISPERSAL

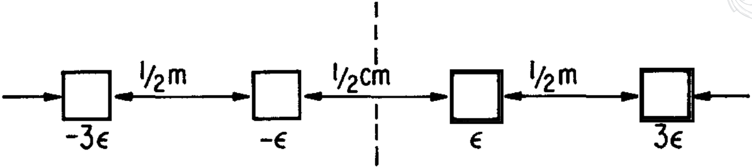


Figure: Stepping stone model with a barrier

from Nagylaki 1976

At each generation,

- the N individuals in $\pm\varepsilon$ are replaced by new individuals
- a proportion $1 - \frac{1+c}{2}m$ of them are the offspring of (uniformly chosen) parents in the same colony,
- a proportion $\frac{1}{2}cm$ are the offspring of parents in $\mp\varepsilon$ and a proportion $\frac{1}{2}m$ come from colony $\pm 3\varepsilon$

EVOLUTION OF ALLELE FREQUENCIES

Individuals are of two types, 0 and 1. Parental type is inherited by the offspring.

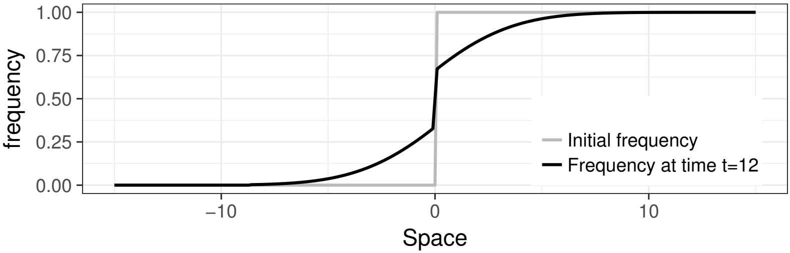


Figure: Evolution of allele frequencies with a barrier

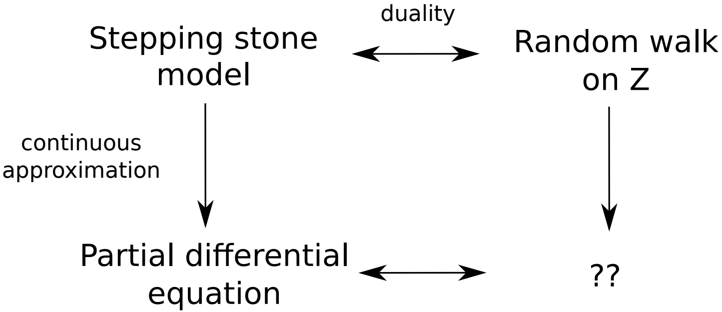
ξ_t : position of the ancestor of a (uniformly) sampled individual t generations in the past = random walk on $\mathbb{Z}$ with transition probabilities given by the migration matrix of the stepping stone model.

$$p(t,x) = \mathbb{P}_x\left(\xi_t \in [0,\infty)\right)$$

APPLICATIONS

Goal : detect barriers to gene flow using genetic data by estimating the age of the most recent common ancestor for different pairs of individuals.

ξ_t : random walk not convenient, no explicit formulas for the law of ξ_t.



MAIN RESULT

For a sequence $(c_n)_{n\in\mathbb{N}}$, let $(\xi_n(t))_{t\geq 0}$ be a random walk on $\mathbb{Z}$ with the corresponding transition probabilities. Set $X_n(t) = \frac{1}{\sqrt{n}}\xi_n(nt)$

Theorem 1

Suppose $\sqrt{n}c_n \underset{n\to\infty}{\longrightarrow} 2\gamma \in [0, +\infty]$, *then*

$$X_n \xrightarrow[n\to\infty]{\textit{sko}} X.$$

The process $(X(t))_{t\geq 0}$ *is (the projection on* $\mathbb{R}$ *of) a Markov process on* $(-\infty, 0^-] \cup [0^+, +\infty)$.

When $\gamma \in (0, \infty)$, we call X partially reflected Brownian motion.

CONSTRUCTION OF PARTIALLY REFLECTED BM 1/2

Start from standard Brownian motion and keep only the excursions outside of $[-\frac{1}{2\gamma}, \frac{1}{2\gamma}]$.

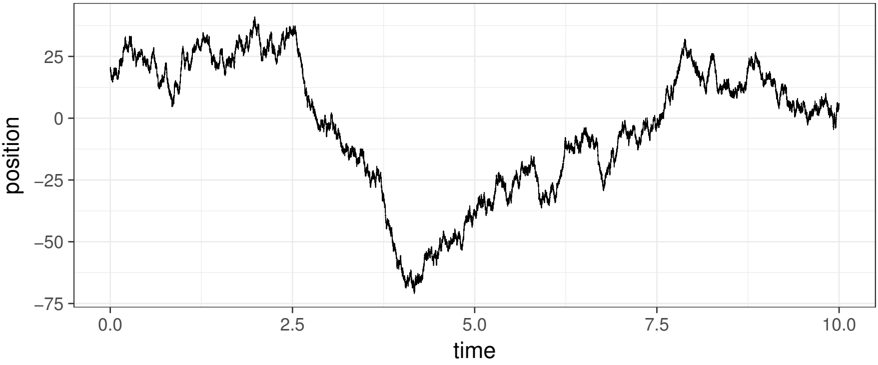


Figure: Speed and scale construction of partially reflected Brownian motion

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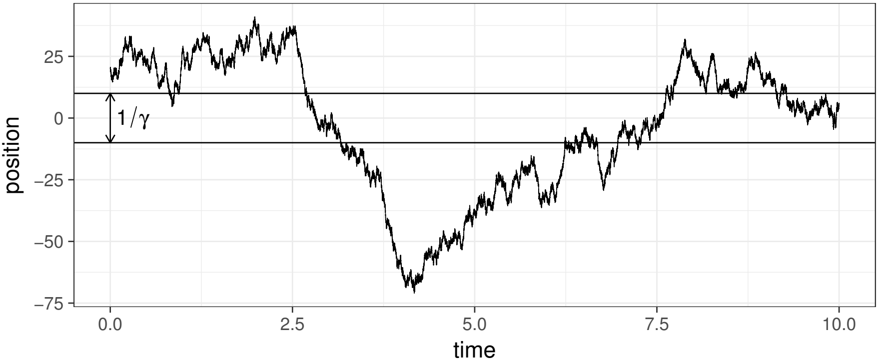

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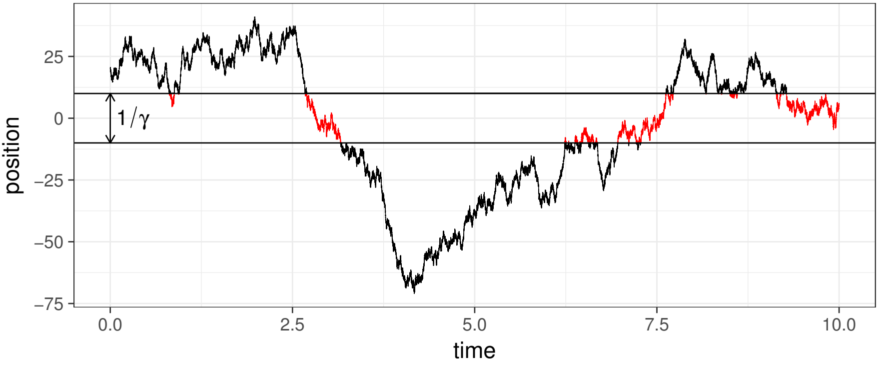

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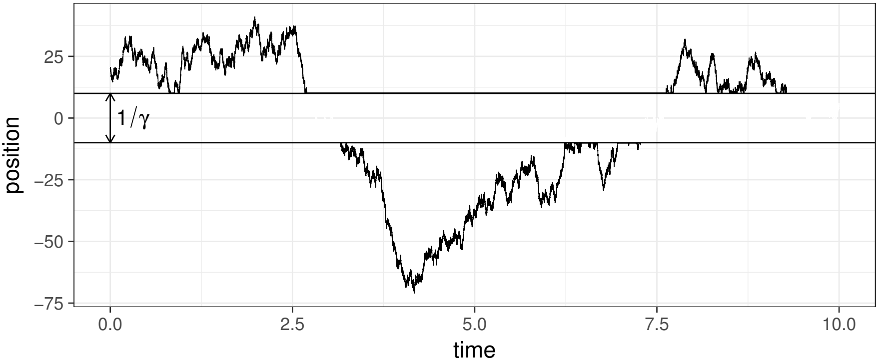

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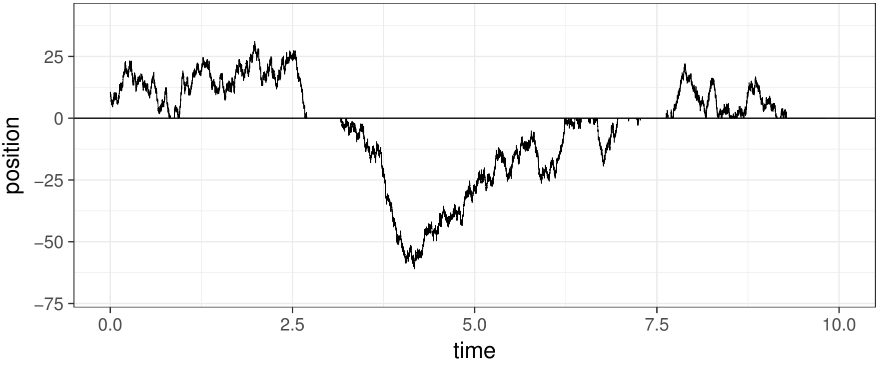

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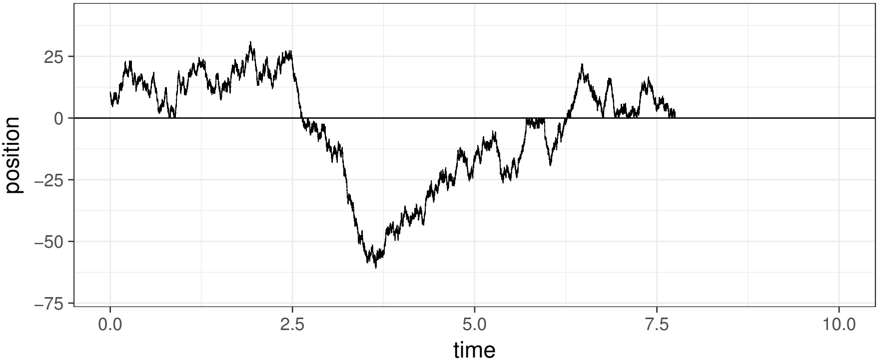
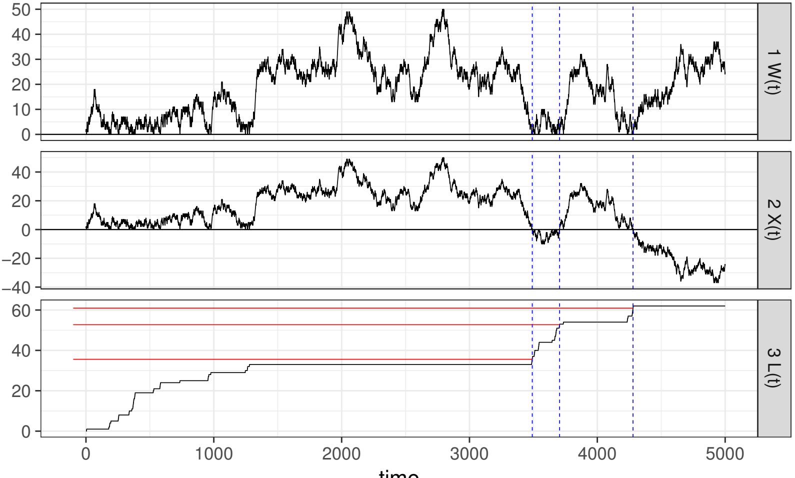


Figure: Speed and scale construction of partially reflected Brownian motion

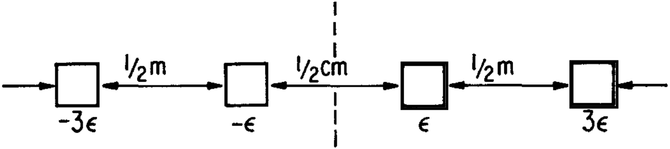
CONSTRUCTION OF PARTIALLY REFLECTED BM 2/2

Start from reflected Brownian motion $(W_t)_{t\geq 0}$, and flip it when its local time at 0 reaches an exponential variable.



SKETCH OF PROOF FOR THE CONVERGENCE RESULT

ξ_t random walk on $\mathbb{Z}$ with transition probabilities



$X_n(t) = \frac{1}{\sqrt{n}}\xi(nt)$

$L_n(t)$: $\frac{1}{\sqrt{n}} \times$ number of visits of X_n to $\{\pm\frac{1}{\sqrt{n}}\}$ up to time t

T_i^n : time of the i-th crossing of $\{\pm\frac{1}{\sqrt{n}}\}$ by X_n

Proof of Theorem 1.

1. $|X_n|$ converges to reflected Brownian motion as $n \to \infty$,
2. $\{L_n(T_{i+1}^n) - L_n(T_i^n), i \geq 0\}$ converges to an iid sequence of $\mathcal{E}(2\gamma)$,
3. the two are asymptotically independent.

□

TRANSITION DENSITIES

We have an explicit formula for the transition densities of $(X_t)_{t\geq 0}$.

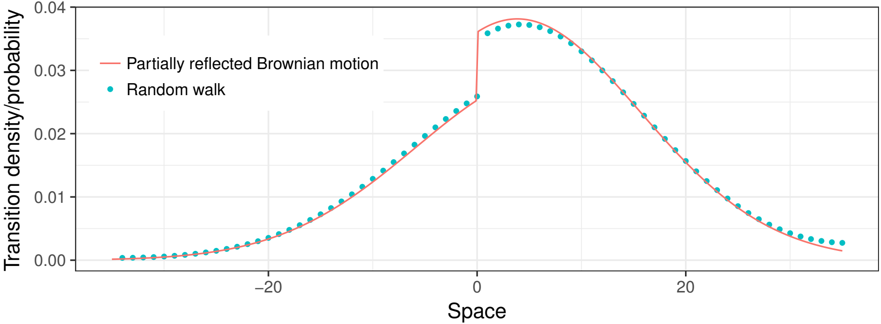
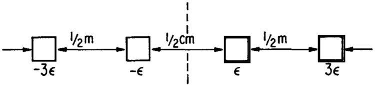


Figure: Comparison of transition probabilities for the random walk and transition densities for partially reflected Brownian motion

SUMMING UP



Stepping stone model

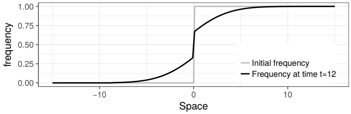
duality

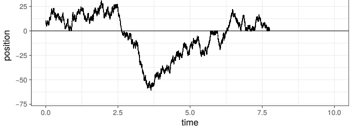
Random walk on Z

continuous approximation

Partial differential equation

Partially reflected Brownian motion





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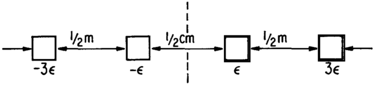
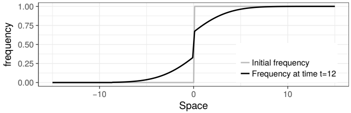
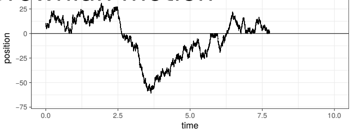


Stepping stone model ←duality→ Random walk on Z

continuous approximation ↓ ↓

Partial differential equation ↔ Partially reflected Brownian motion





$$p(t,x) = \mathbb{P}_x\left(X_t \in [0^+, +\infty)\right)$$

Thank you for your attention !