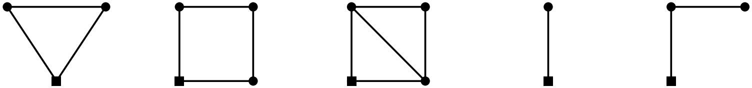
Scaling limits for graphs constructed through a generalised Rémy algorithm

Delphin Sénizergues

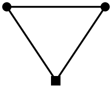
LAGA, Université Paris 13

May 3, 2018

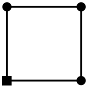
Let $(G_n)_{n\geq 1}$ be a sequence of connected discrete pointed graphs. We are going to define a sequence of graphs $(H_n)_{n\geq 1}$ recursively.



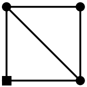
G_1 G_2 G_3 G_4 G_5



G_1

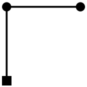


G_2

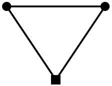


G_3

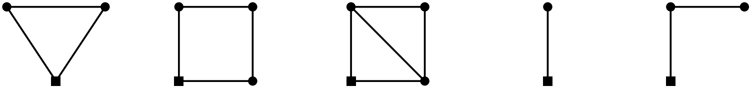
G_4



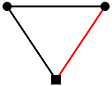
G_5

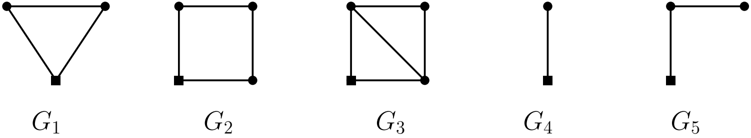


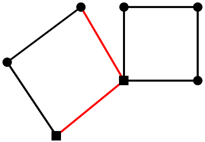
H_1

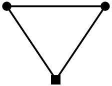


G_1 G_2 G_3 G_4 G_5

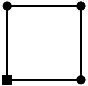



G_1
G_2
G_3
G_4
G_5

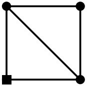




G_1

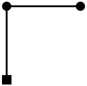


G_2

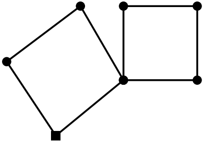


G_3

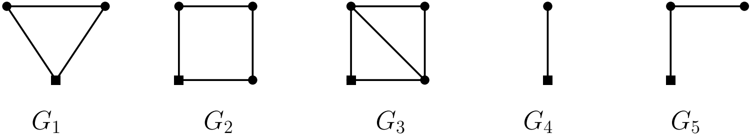
G_4

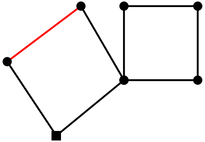


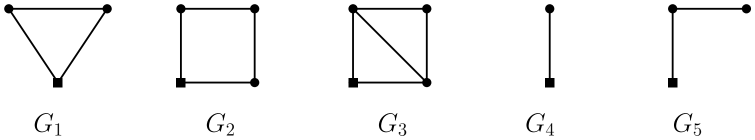
G_5

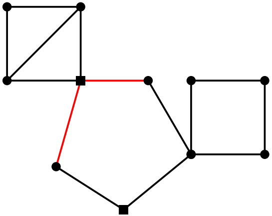


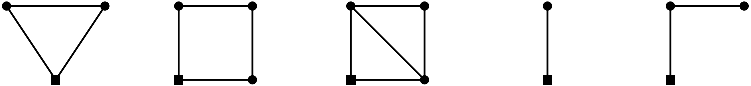
H_2


G_1
G_2
G_3
G_4
G_5

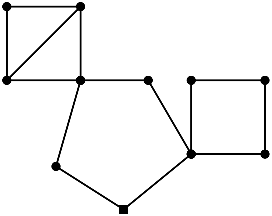



G_1
G_2
G_3
G_4
G_5

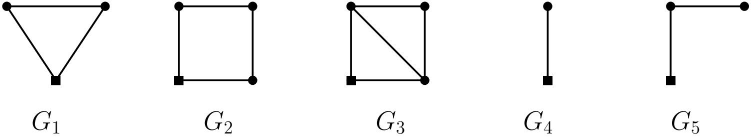


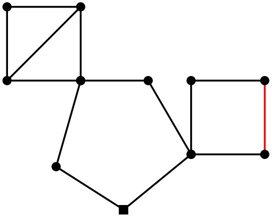


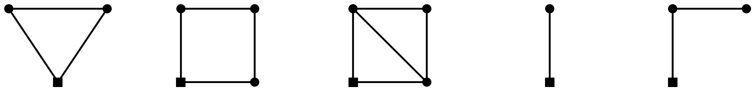
G_1 G_2 G_3 G_4 G_5



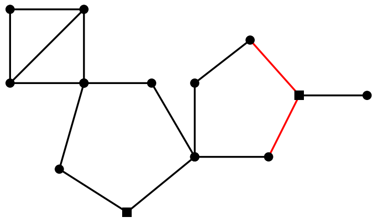
H_3

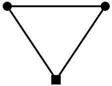

G_1
G_2
G_3
G_4
G_5

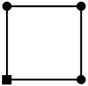


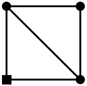


G_1 G_2 G_3 G_4 G_5

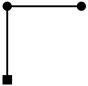


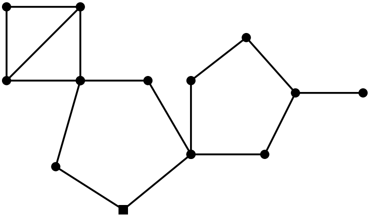

G_1

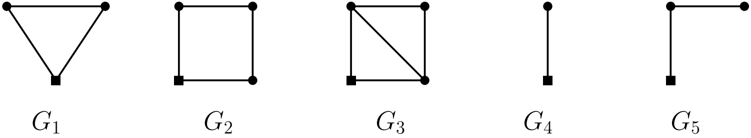

G_2

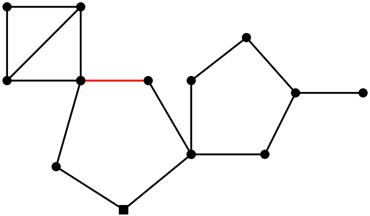

G_3

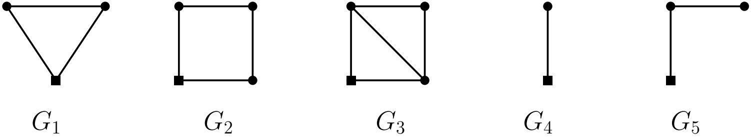

G_4

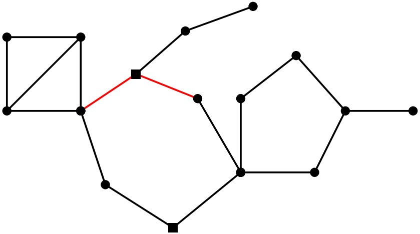

G_5

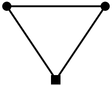

H_4


G_1
G_2
G_3
G_4
G_5

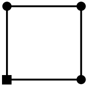



G_1
G_2
G_3
G_4
G_5

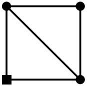




G_1

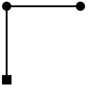


G_2

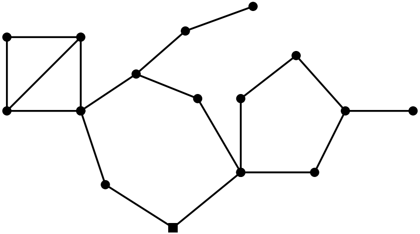


G_3

G_4



G_5



H_5

If H_n is already defined, then

If H_n is already defined, then

take an edge uniformly at random on H_n (independently on everything else),

If H_n is already defined, then

- take an edge uniformly at random on H_n (independently on everything else),
- split this edge into 2 edges with a common endpoint,

If H_n is already defined, then

take an edge uniformly at random on H_n (independently on everything else),

split this edge into 2 edges with a common endpoint,

add a copy of the graph G_{n+1} such that the newly created vertex corresponds to the root of G_{n+1}.

If H_n is already defined, then

- take an edge uniformly at random on H_n (independently on everything else),
- split this edge into 2 edges with a common endpoint,
- add a copy of the graph G_{n+1} such that the newly created vertex corresponds to the root of G_{n+1}.

This defines H_{n+1}. We want to understand the behaviour of H_n when n is very large.

A non-empty finite connected graph G can be seen as a compact measured metric space, by endowing its set of vertices $V(G)$ with the graph distance d_{gr} and the uniform measure μ_{unif}.

A non-empty finite connected graph G can be seen as a compact measured metric space, by endowing its set of vertices $V(G)$ with the graph distance d_{gr} and the uniform measure μ_{unif}.
If $x, y \in V(G)$, their graph distance $\mathrm{d}_{\mathrm{gr}}(x, y)$ is the length of the shortest path in the graph from x to y.

We see $(V(G), \mathrm{d}_{\mathrm{gr}}, \mu_{n,\mathrm{unif}})$ as an element of

$$\mathbb{M} = \{\text{compact metric spaces endowed with a Borel probability measure, seen up to measure preserving isometry}\}.$$

Two measured metric spaces (X, d, μ) and (X', d', μ') are identified if and only if there exists a bijective isometry $\phi : X \to X'$ such that $\phi_* \mu = \mu'$. Let us define a topology on $\mathbb{M}$.

Let (E, d) be a metric space, and let A and B be non-empty compact subsets of E. The Hausdorff distance between A and B is defined as

$$\mathrm{d}_{\mathrm{H}}^{E}(A, B) = \inf \left\{ \epsilon > 0 \,\middle|\, A \subset B^{(\epsilon)}, \quad B \subset A^{(\epsilon)} \right\}.$$

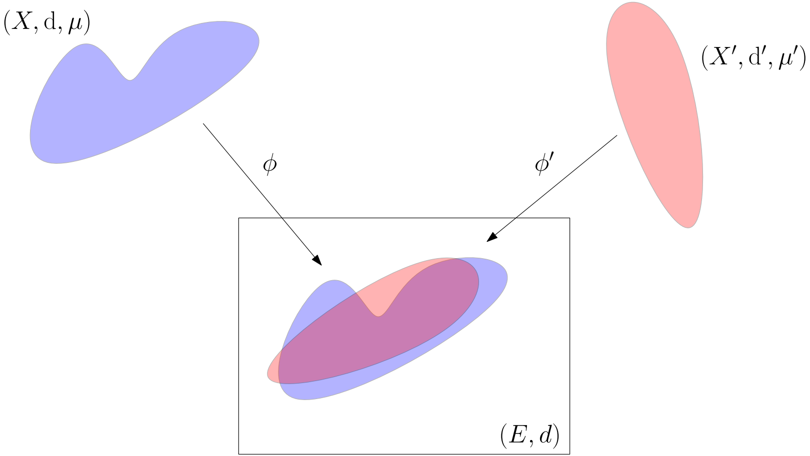
The ϵ-fattening $C^{(\epsilon)}$ of the set C is defined as

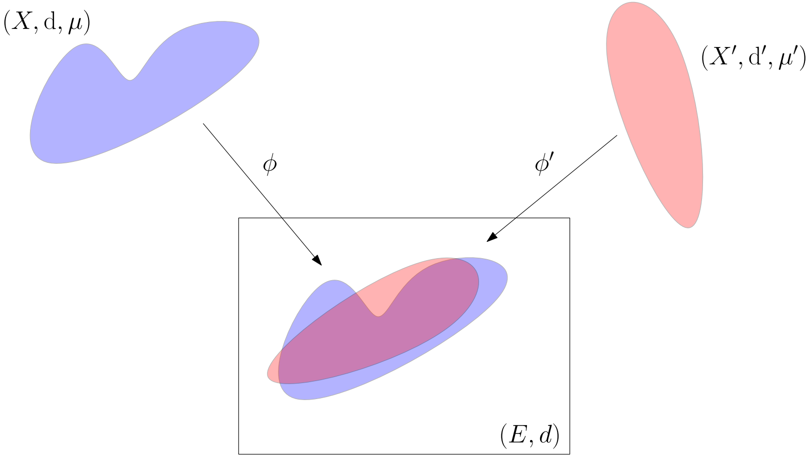
$$C^{(\epsilon)} := \{ x \in E \mid d(x, C) < \epsilon \}.$$

We denote $\mathcal{P}(E)$ the set of Borel probability measures on E. For any two $\mu, \nu \in \mathcal{P}(E)$, we can define their Lévy-Prokhorov distance as,

$$\mathrm{d}_{\mathrm{LP}}^{E}(\mu, \nu) \\ = \inf\left\{\epsilon > 0 \;\middle|\; \forall F \in \mathcal{B}(E), \mu(F) \leq \nu(F^{(\epsilon)}) + \epsilon, \nu(F) \leq \mu(F^{(\epsilon)}) + \epsilon\right\}$$


(X, d, μ)
(X', d', μ')


(X, d, μ)
(X', d', μ')
ϕ
ϕ'
(E, d)



$$\mathrm{d}_{\mathrm{GHP}}((X,d,\mu),(X',d',\mu'))$$
$$= \inf_{(E,d),\phi:X\to E,\phi':X'\to E} \left\{ \mathrm{d}_{\mathrm{H}}^{E}(\phi(X),\phi'(X')) \vee \mathrm{d}_{\mathrm{LP}}^{E}(\phi_*\mu,(\phi')_*\mu') \right\}.$$

The function $\mathrm{d}_{\mathrm{GHP}}$ is a distance on $\mathbb{M}$ which makes it a Polish space (separable and complete metric space).

Denote $\mathbf{a} = (a_n)_{n\geq 1} = (|E(G_n)|)_{n\geq 1}$ the sequence corresponding to the number of edges in $(G_n)_{n\geq 1}$. Suppose that

$$\sum_{i=1}^{n} a_i = cn + r_n, \text{ with } \sum_{n=1}^{\infty} \frac{r_n}{n^2} < \infty.$$

Theorem (S. 2017+)

We have the following convergence

$$\left(H_n, n^{-\frac{1}{c+1}} \cdot \mathrm{d}_{\mathrm{gr}}, \mu_{n,\mathrm{unif}}\right) \underset{n\to\infty}{\longrightarrow} (\mathcal{H}, \mathrm{d}, \mu),$$

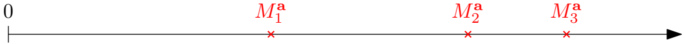
almost surely in Gromov-Hausdorff-Prokhorov topology.

We can describe the space $(\mathcal{H}, \mathrm{d}, \mu)$ using a *line-breaking construction*.

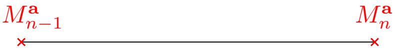
We can describe the space $(\mathcal{H}, \mathrm{d}, \mu)$ using a *line-breaking construction*. We start with $(M_n^{\mathbf{a}})_{n \geq 1}$, a positive increasing (possibly time-inhomogeneous) Markov chain the distribution of which depends on $\mathbf{a}$.

We can describe the space $(\mathcal{H}, \mathrm{d}, \mu)$ using a *line-breaking construction*. We start with $(M_n^{\mathbf{a}})_{n \geq 1}$, a positive increasing (possibly time-inhomogeneous) Markov chain the distribution of which depends on **a**. "Break" the half-line $\mathbb{R}_+$ at the values taken by the chain.

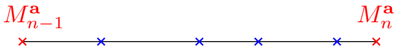
We can describe the space $(\mathcal{H}, \mathrm{d}, \mu)$ using a *line-breaking construction*. We start with $(M_n^{\mathbf{a}})_{n\geq 1}$, a positive increasing (possibly time-inhomogeneous) Markov chain the distribution of which depends on **a**. "Break" the half-line $\mathbb{R}_+$ at the values taken by the chain.



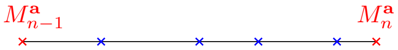
For every $n \geq 1$, throw $(a_n - 1)$ i.i.d. uniform random points on the segment $[M^{\mathbf{a}}_{n-1}, M^{\mathbf{a}}_n]$.



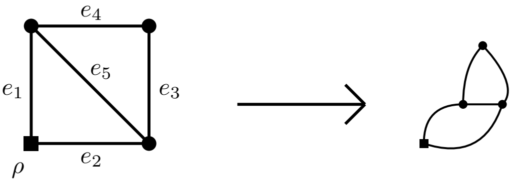
For every $n \geq 1$, throw $(a_n - 1)$ i.i.d. uniform random points on the segment $[M^{\mathbf{a}}_{n-1}, M^{\mathbf{a}}_n]$.



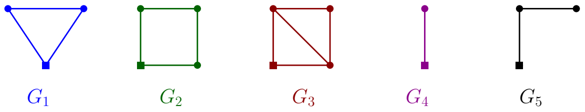
For every $n \geq 1$, throw $(a_n - 1)$ i.i.d. uniform random points on the segment $[M^{\mathbf{a}}_{n-1}, M^{\mathbf{a}}_n]$. It breaks the segment into a_n bits,



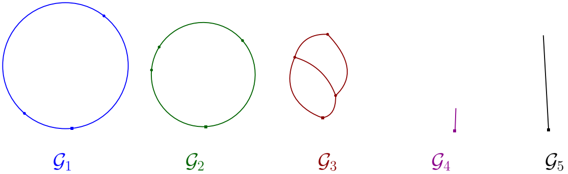
For every $n \geq 1$, throw $(a_n - 1)$ i.i.d. uniform random points on the segment $[M^{\mathbf{a}}_{n-1}, M^{\mathbf{a}}_n]$. It breaks the segment into a_n bits, For every $n \geq 1$, throw $(a_n - 1)$ i.i.d. uniform random points on the segment $[M^{\mathbf{a}}_{n-1}, M^{\mathbf{a}}_n]$. It breaks the segment into a_n bits, that we glue together to construct $\mathcal{G}_n$, a continuous version of G_n.

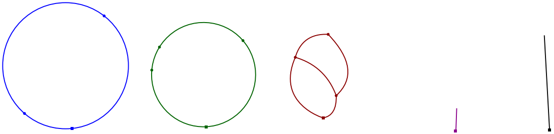


G_n $\mathcal{G}_n$

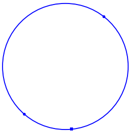

G_1
G_2
G_3
G_4
G_5

A line breaking construction

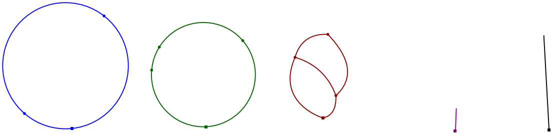




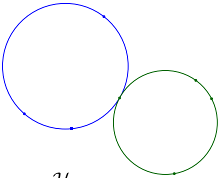
$\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_3$ $\mathcal{G}_4$ $\mathcal{G}_5$



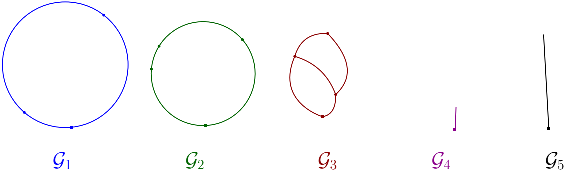
$\mathcal{H}_1$

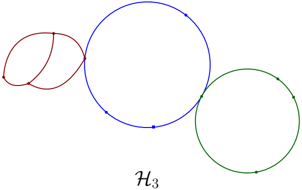


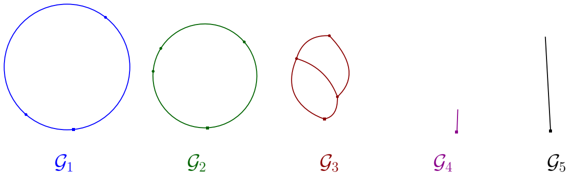
$\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_3$ $\mathcal{G}_4$ $\mathcal{G}_5$

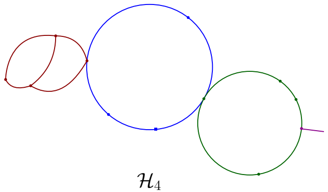


$\mathcal{H}_2$

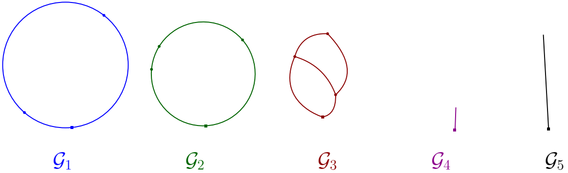

$\mathcal{G}_1$
$\mathcal{G}_2$
$\mathcal{G}_3$
$\mathcal{G}_4$
$\mathcal{G}_5$

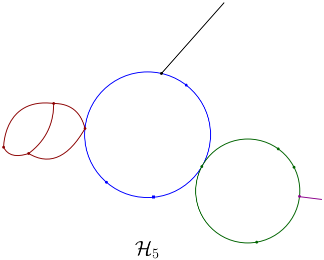

$\mathcal{H}_3$


$\mathcal{G}_1$
$\mathcal{G}_2$
$\mathcal{G}_3$
$\mathcal{G}_4$
$\mathcal{G}_5$


$\mathcal{H}_4$

A line breaking construction





Start with $\mathcal{H}_1 = \mathcal{G}_1$.

Start with $\mathcal{H}_1 = \mathcal{G}_1$. If $\mathcal{H}_n$ is already defined:

Sample a uniform point on the length of $\mathcal{H}_n$,

Start with $\mathcal{H}_1 = \mathcal{G}_1$. If $\mathcal{H}_n$ is already defined:

Sample a uniform point on the length of $\mathcal{H}_n$,

Glue a copy of $\mathcal{G}_{n+1}$ by identifying its root to the chosen point,

Start with $\mathcal{H}_1 = \mathcal{G}_1$. If $\mathcal{H}_n$ is already defined:

Sample a uniform point on the length of $\mathcal{H}_n$,

Glue a copy of $\mathcal{G}_{n+1}$ by identifying its root to the chosen point,

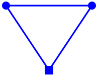
Then $\mathcal{H}_{n+1}$ is defined.

Start with $\mathcal{H}_1 = \mathcal{G}_1$. If $\mathcal{H}_n$ is already defined:

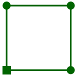
- Sample a uniform point on the length of $\mathcal{H}_n$,
- Glue a copy of $\mathcal{G}_{n+1}$ by identifying its root to the chosen point,
- Then $\mathcal{H}_{n+1}$ is defined.

$\mathcal{H}$ is obtained as:

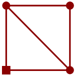
$$\mathcal{H} = \overline{\bigcup_{n \geq 1} \mathcal{H}_n}.$$



G_1

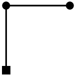


G_2

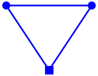


G_3

G_4

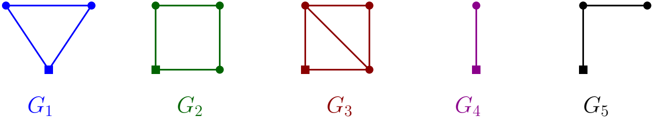


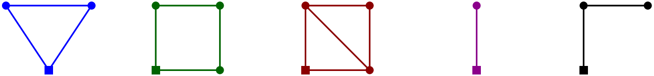
G_5



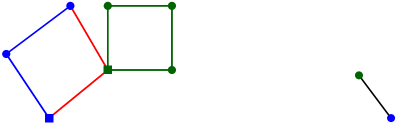
H_1

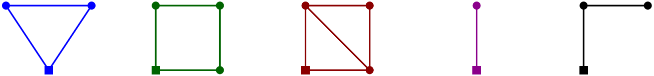
T_1


G_1
G_2
G_3
G_4
G_5

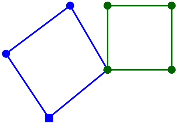


G_1 G_2 G_3 G_4 G_5

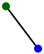




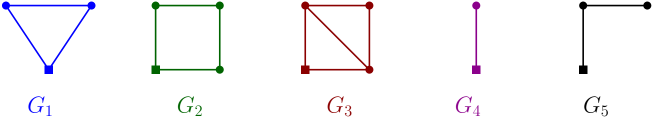
G_1 G_2 G_3 G_4 G_5

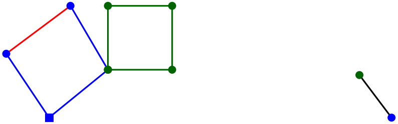


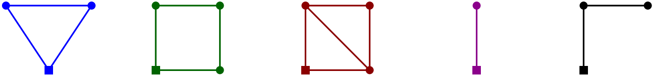
H_2



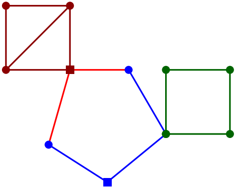
T_2

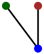

G_1
G_2
G_3
G_4
G_5

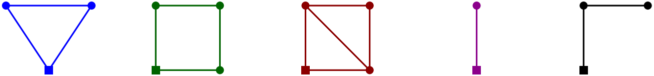




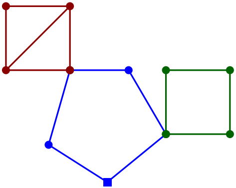
G_1 G_2 G_3 G_4 G_5



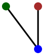




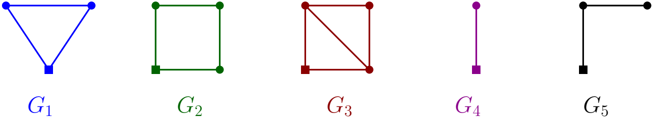
G_1 G_2 G_3 G_4 G_5

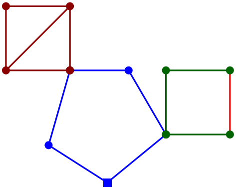


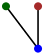
H_3

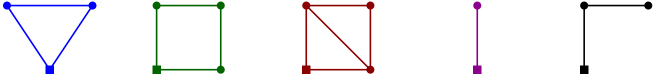


T_3

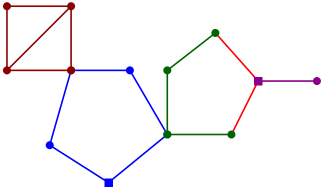

G_1
G_2
G_3
G_4
G_5

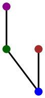


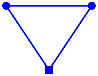




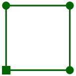
G_1 G_2 G_3 G_4 G_5



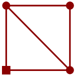




G_1

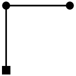


G_2

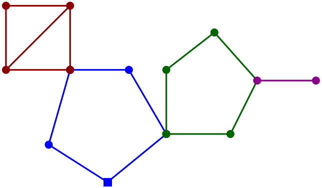


G_3

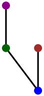
G_4



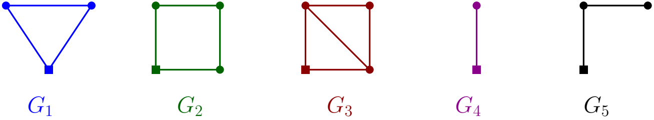
G_5

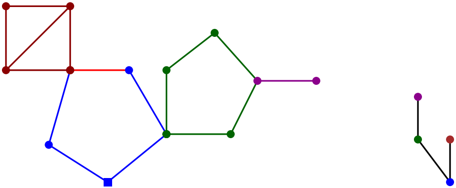


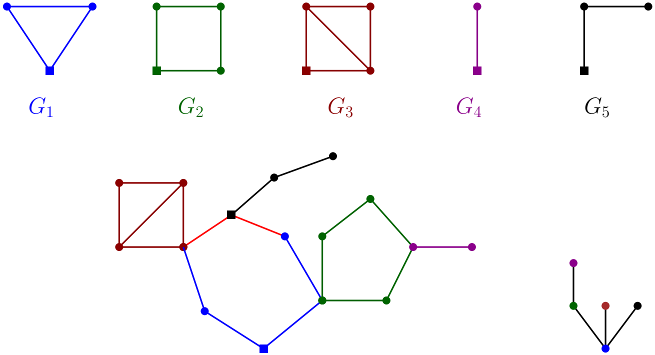
H_4

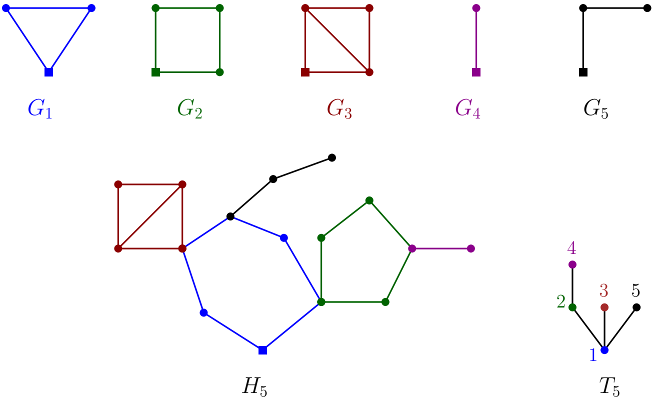


T_4


G_1
G_2
G_3
G_4
G_5




G_1
G_2
G_3
G_4
G_5


G_1
G_2
G_3
G_4
G_5
H_5
4
3
5
2
1
T_5

Thank you for your attention !